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Applications of Differential Transform Method To Initial Value Problems

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ABSTRACT: *In this article the Differential Transform method is employed for obtaining solutions for initial value problems. This method gives the series of solutions which can be easily converted to exact ones. The differential transform method was successfully applied to initial value problems. The findings of the study has demonstrated that the method is easy , effective and flexible. The results of the differential transform method is in good agreement with those obtained by using the already existing ones. The proposed method is promising to a broad class of linear and nonlinear problems.*

Keywords: *Differential Transform, Nonlinear, Initial value problems*

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I. INTRODUCTION

Nonlinear phenomena have important effects in applied mathematics, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear differential equations [3,4,10]. A variety of numerical and analytical methods have been developed to obtain accurate approximate and analytic solutions for the problems in the literature [3,7,8,10,11,12]. The classical Taylor's series method is one of the earliest analytic techniques to many problems, especially ordinary differential equations. However, since it requires a lot of symbolic calculation for the derivatives of functions, it takes a lot of computational time for higher derivatives. Here, we introduce the update version of the Taylor series method which is called the differential transform method (DTM)[4,5]. The (DTM) is the method to determine the coefficients of the Taylor series of the function by solving the induced recursive equation from the given differential equation. The basic idea of the (DTM) was introduced by Zhou [5]. In what follows we introduce a few notations for the (DTM).

II. THE DIFFERENTIAL TRANSFORM METHOD

Suppose that the solution $u(x;t)$ is analytic at $(X;Y)$, then the solution $u(x;t)$ can be represented by the Taylor series[1].

$$u(x,t) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k_1! \dots k_n!\, h!} \left[\frac{\partial^{k_1+\dots+k_n+h} u(\tilde{x},\tilde{t})}{\partial x^{k_1} \dots \partial x^{k_n} \partial x^{h} u(\tilde{x},\tilde{t})} \right]$$

$$\left(\prod_{i=1}^{n} (x_i - \tilde{x}_i)^{k_i}\right)(t-\tilde{t})^{h} \quad (1)$$

Definition 1.1

Let us define the $(n+1)$ dimensional differential transform $U(\tilde{k},\tilde{h})$ by

$$U(x,t) = \frac{1}{k_1! \dots k_n!\, h!} \left[\frac{\partial^{k_1+\dots+k_n+h} u(\tilde{x},\tilde{t})}{\partial x^{k_1} \dots \partial x^{k_n} \partial x^{h} u(\tilde{x},\tilde{t})} \right] \quad (2)$$

Definition 1.2

The differential inverse transform of $U(x,t)$ is define by $u(x,t)$ of the form in (1). Thus $u(x,t)$ can be written by:

$$u(x,t) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \sum_{h=0}^{\infty} U(k,h) \left(\prod_{i=1}^{n} (x_i - \tilde{x}_i)^{k_i} \right) (t-\tilde{t})^{\,h} \qquad (3)$$

An arbitrary function $f(x)$ can be expanded in Taylor series about a point $x = 0$ as:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left\{ \frac{d^k f}{dx^k} \right\}_{x=0} \qquad (4)$$

The differential inverse transform of $u(x,t)$ is define by:

$$F(x) = \frac{1}{k!} \left\{ \frac{d^k f(x)}{dx^k} \right\}_{x=0} \qquad (5)$$

Then the inverse differential transform is:

$$F(x) = \sum_{k=0}^{\infty} x^k \; F(k) \qquad (6)$$

3 The fundamental operation of Differential Transformation Method [2]:

(3.1) If $y(x) = g(x) \pm h(x)$ then $Y(k) = G(k) \pm H(k)$

$$\begin{aligned} F(k) &= \frac{1}{k!}\left\{\frac{d^k f}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k y(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k\big(g(x) \pm h(x)\big)}{dx^k}\right\}_{x=0} \\ &= \frac{1}{k!}\left\{\frac{d^k g(x)}{dx^k} \pm \frac{d^k h(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k g(x)}{dx^k}\right\}_{x=0} \pm \frac{1}{k!}\left\{\frac{d^k h(x)}{dx^k}\right\}_{x=0} \\ &= G(x) \pm H(x) \end{aligned}$$

(3.2) If $y(x) = \alpha g(x)$ then $Y(k) = \alpha G(x)$.

(3.3)If $y(x) = \frac{dg(x)}{dx}$ then Y(k)=(k+1)G(k),

$$\begin{aligned} Y(k) &= \frac{1}{k!}\left\{\frac{d^k y(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k}{dx^k}\left(\frac{d^k g(x)}{dx^k}\right)\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^{k+1} g(x)}{dx^{k+1}}\right\}_{x=0} \\ &= \frac{1}{k!}\,(k+1)!\,G(k+1) = (k+1)G(k) \end{aligned}$$

(3.4) If y(x)=$\frac{d^2 g(x)}{dx^2}$ then $Y(k) = (k+1)(k+2)G(k+1)$

(3.5) If y(x)=$\frac{d^m g(x)}{dx^m}$ then $Y(k) = (k+1)(k+2)\dots(k+m)G(k+m)$

(3.6) If $y(x) = 1$ then $Y(k) = \delta(k)$

(3.7) If $y(x) = x$ then $Y(k) = \delta(k-1)$

(3.8) If $y(x) = x^m$ then $Y(k) = \delta(k-m) = \begin{cases} 1 \ if \ k = m \\ 0 \ if \ k \neq m \end{cases}$

(3.9) If $y(x) = g(x)h(x)$ then $Y(k) = \sum_{m=0}^{k} H(k)G(k-m)$

(3.10) If $y(x) = e^{\alpha x}$ then $Y(k) = \frac{\alpha^k}{k!}$

(3.11) If $y(x) = (1+x)^m$ then $Y(k) = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$

(3.12) If $y(x) = \sin(wx + \alpha)$, then $Y(k) = \frac{w^k}{k!}\sin(k\pi + \alpha)$ where w and α are constants.

(3.13) If $y(x) = \cos(wx + \alpha)$, then $Y(k) = \frac{w^k}{k!}\cos(k\pi + \alpha)$ where w and α are constants.

IV. APPLICATIONS

In this section , we apply the (DTM) to some ordinary differential equations and then to Voltera equation.

4.1 Problem 1

Consider the following initial value problem [6]

$$\begin{cases} x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2 \\ y(1) = \dfrac{4}{3} \\ y'(1) = \dfrac{5}{3} \end{cases} \tag{7}$$

By using the transformation $x = e^t$ the problem is converted to:

$$\begin{cases} \dfrac{d^2y}{dt^2} = y + e^{2t} \\ y(0) = \dfrac{4}{3} \\ \acute{y}(0) = \dfrac{5}{3} \end{cases} \tag{8}$$

Using the (DTM) we have:

$$(k+1)(k+2)Y(k+2) = Y(k) + \frac{2^k}{k!} \tag{9}$$

Thus:

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[Y(k) + \frac{2^k}{k!}\right] \tag{10}$$

with the following conditions:

$$Y(0) = \frac{4}{3} \qquad , Y(1) = \frac{5}{3} \tag{11}$$

when $k = 0$ then $Y(2) = \frac{7}{6}$, when $k = 1 \rightarrow Y(3) = \frac{11}{18}$,when $k = 2 \rightarrow Y(4) = \frac{19}{72}$,...

The solution is:

$$y(t) = \sum_{k=0}^{\infty} t^k Y(k) = \frac{4}{3} + \frac{5}{3}t + \frac{7}{6}t^2 + \frac{11}{18}t^3 + \frac{47}{216}t^4 + \cdots \tag{12}$$

This can be written as:

$$\begin{aligned} y(t) &= 1 + \frac{1}{3} + t + \frac{2}{3}t + \frac{1}{2}t^2 + \frac{4}{2}t^2 + \frac{1}{6}t^3 + \frac{8}{18}t^3 + \frac{1}{24}t^4 + \frac{16}{72}t^4 + \cdots \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{1}{3}\left(1 + (2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \cdots\right) \end{aligned}$$

$$= e^t + \frac{1}{3}e^{2t} \tag{13}$$

The required answer is:

$$y(x) = x + \frac{x^2}{3}$$

4.2 Problem 2

We, next consider the following nonlinear problem [6]:

$$\begin{cases} x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y + y^2 = (x\ln(x))^2 \\ y(1) = 0 \\ y'(1) = 1 \end{cases} \tag{14}$$

using the transformation $x = e^t$ the problem converted to:

$$\begin{gathered} \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y + y^2 = t^2e^{2t} \\ y(0) = 0 \\ y'(0) = 1 \end{gathered} \tag{15}$$

Using the (DTM) we have:

$$(k+1)(k+2)Y(k+2) - 2(k+1)Y(k+1) + Y(k) + \sum_{m=0}^{k} Y(k)Y(k-m) = \sum_{m=0}^{k} \delta(m-2)\frac{2^{k-m}}{(k-m)!} \tag{16}$$

or

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left(2(k+1)Y(k+1) - Y(k) - \sum_{m=0}^{k} Y(k)Y(k-m) + \sum_{m=0}^{k} \delta(m-2)\frac{2^{k-m}}{(k-m)!}\right) \tag{17}$$

with the following conditions

$$Y(0) = 0 \;\; and \; Y(1) = 1 \tag{18}$$

If $k = 0$ then $Y(2) = 1$, when $k = 1 \to Y(3) = \frac{1}{2}$ when $k = 2 \to Y(4) = \frac{1}{6}$, ...
Thus:

$$y(t) = \sum_{k=0}^{\infty} t^k Y(k) \quad = 0 + t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \cdots$$

$$= t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right) = te^t \tag{19}$$

Therefore the required solution is:

$$y(x) = x\ln(x) \tag{20}$$

For the next problem, we need the following theorem:

4.3 Problem 3

Consider the following initial value problem [6]

$$\begin{cases} x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^5 \\ y(1) = \dfrac{1}{2} \\ y'(1) = \dfrac{3}{2} \end{cases} \qquad (21)$$

By using the transformation $x = e^t$ the problem is converted to:

$$\begin{cases} \dfrac{d^2y}{dt^2} = 5\dfrac{d^2y}{dx^2} - 6y + e^{5t} \\ y(0) = \dfrac{1}{2} \\ \acute{y}(0) = \dfrac{3}{2} \end{cases} \qquad (22)$$

Using the (DTM) we have:

$$(k+1)(k+2)Y(k+2) = 5(k+1)Y(k+1) - 6Y(k) + \frac{5^k}{k!} \qquad (23)$$

Thus:

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left(5(k+1)Y(k+1) - 6Y(k) + \frac{5^k}{k!}\right)$$

with the following conditions:

$$Y(0) = \frac{1}{2} \qquad , Y(1) = \frac{3}{2} \qquad (24)$$

Consequently:

If $k = 0$ then $Y(2) = \frac{11}{4}$, when $k = 1 \to Y(3) = \frac{47}{12}$ when $k = 2 \to Y(4) = \frac{219}{48}$, ...

Thus:

$$y(t) = \sum_{k=0}^{\infty} t^k U(k) = \frac{1}{2} + \frac{3}{2}t + \frac{11}{4}t^2 + \frac{47}{12}t^3 + \frac{219}{48}t^4 + \cdots \qquad (25)$$

or

$$\begin{aligned} y(t) &= \frac{1}{3}\left(1 + (2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \cdots\right) \\ &\quad + \frac{1}{6}\left(1 + (5t) + \frac{(5t)^2}{2!} + \frac{(5t)^3}{3!} + \frac{(5t)^3}{4!}\right) \\ &= \frac{1}{3}e^{2t} + \frac{1}{6}e^{5t} \end{aligned} \qquad (26)$$

Therefore the required solution is:

$$y(x) = \frac{1}{3}x^2 + \frac{1}{6}x^5 \tag{27}$$

4.4 Problem 4

Consider the following initial value problem [6]

$$\begin{cases} x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y^2 = \left(ln(x)\right)^3 \\ y(1) = 1 \\ y'(1) = 2 \end{cases} \tag{28}$$

By using the transformation $x = e^t$ the problem is converted to:

$$\begin{cases} \dfrac{d^2y}{dt^2} = -2\dfrac{dy}{dt} - y^2 + t^3 \\ y(0) = 1 \\ \acute{y}(0) = 2 \end{cases} \tag{29}$$

Using the (DTM) we have:

$$(k+1)(k+2)Y(k+2) = -2(k+1)Y(k+1) - \sum_{m=0}^{k} Y(k)Y(k-m) + 6\delta(k-3) \tag{30}$$

Thus:

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left(-2(k+1)Y(k+1) - \sum_{m=0}^{k} Y(k)Y(k-m) \ + \delta(k-3)\right) \tag{31}$$

with the following conditions:

$$Y(0) = 1 \quad ,Y(1) = 2 \tag{32}$$

Consequently:

If $k = 0$ then $Y(2) = \frac{-5}{2}$, when $k = 1 \rightarrow Y(3) = \frac{2}{3}$ when $k = 2 \rightarrow Y(4) = -\frac{11}{48}$, if when $k = 2 \rightarrow Y(4) = \frac{37}{360}$,

Thus:

$$y(t) = \sum_{k=0}^{\infty} t^k U(k) = 1 + 2t - \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{11}{48}t^4 + \frac{37}{360}t^5 \ + \cdots \tag{33}$$

The required solution is:

$$y(x) = 1 + 2\ln(x) - \frac{5}{2}(\ln(x))^2 + \frac{2}{3}(\ln(x))^3 - \frac{11}{48}(\ln(x))^4 + \frac{37}{360}(\ln(x))^5 + \cdots \quad (35)$$

V. CONCLUSION

The observations of the present study have shown that the (DTM) is easy to implement and effective. As a result, the conclusion comes through this work, is that the Differential Transform Method can be applied to a wide class of differential equations, due to the efficiency in the application to get the possible results.

REFERENCES

[1]. Ayaz.F, Solutions of system of differential equation by Differential Transform Method.Appl.Math.comput 2004.pp147-547-567.
[2]. G.Adoman.R.Rack, On linear and nonlinear integro-differential equations. J. Math. Anal. Appl.113(1)(1986), pp. 117-120.
[3]. Kaya.N.T. Comparing numerical methods for solutions of ordinary differential equations, Appl.Math. Lett, 17(2004)323-328.
[4]. Liu.H and Y.Song, Differential Transform Method applied to higher index differential- algebraic equations. Appl.Math. Comput,2007,184-748-753.
[5]. Zhou.J.K. Differential transformation Method and applications for electrical circuits, Huazhong University press, Wuhan, China, (1986)
[6]. Onur Kiymaz, An Algorithm for Solving Initial Value Problems, using Laplace Adomian Decomposition Method. Applied Mathematical Sciences, Vol. 3, 2009, no. 30, 1453 - 1459
[7]. E. Hesameddini, H. Latifizadeh. A new vision of the He's homotopy_ perturbation method. International Journal of Nonlinear Sciences and Numerical Simulation. 2009.
[8]. E. Hesameddini, H. Latifizadeh. Reconstruction of variational iteration algorithms using the Laplace transform. International Journal of Nonlinear Sciences and Numerical Simulation. 2009.
[9]. S.T. Mohyud-Din, M.A. Noor, K.I. Noor. Some relatively new techniques for nonlinear problems. Math. Porb.Eng. Article ID 234849, 25 pages, doi: 10.1155/2009/234849, 2009.
[10]. S.T. Mohyud-Din, A. Yildirim. Variational_ iteration method for solving Klein- Gordon equations. Journal of Applied Mathematics, Statistics and Informatics.2010.
[11]. Jagdev Singh, Devendra_ Kumar and Sushila Rathore, Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinear
[12]. Klein- Gordon Equations, Journal of Information and Computing Science,ISSN 1746-765 England, UK,Vol. 7, No. 2, 2012, pp. 131-139.
[13]. A. Yildirim. An Algorithm for Solving the Fractional Nonlinear Schröndinger Equation by Means of the Homotopy Perturbation Method. International Journal of Nonlinear Science and Numerical Simulation. 2009.

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