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Analysis of Fractional Order Prey - Predator Interactions

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ABSTRACT: *The dynamical behavior of a Fractional order Prey - Predator model is investigated in this paper. The equilibrium points are computed and stability of the equilibrium points is analyzed. The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed and they exhibit rich dynamics of the fractional model.*

Keywords – *Fractional Order, differential equations, Prey - Predator, stability.*

I. INTRODUCTION

Fractional order integral and derivative operators have found several applications in large areas of research during the last decade. The concept of fractional calculus was raised in the year 1695 by Marquis de L'Hopital to Gottfried Wilhelm Leibniz regarding solution of non-integer order derivative. On September 30th 1695, Leibniz replied to L'Hopital "This is an apparent paradox from which one day, useful consequences will be drawn". Between 1695and 1819 several mathematicians (Euler in 1730, Lagrange in 1772, Laplace in 1812, and soon..) mentioned it. The question raised in 1695 was only partly answered 124 years later in 1819, by S. F. Lacroix. The real journey of development of fractional calculus started in1974 when the first monograph on fractional calculus was published by academic press [7]. Recently theory of fractional differential and its applications has attracted much attention.

II. FRACTIONAL DERIVATIVES AND INTEGRALS

In this section, we present important definitions of fractional calculus which arise as natural generalization of results from calculus [2, 5].

Definition 1. The Riemann - Liouville fractional Integral of order$0 \leq \alpha \leq 1$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{(\alpha-1)} f(u)\, du, t > 0$$

Definition2. The Riemann - Liouville fractional derivative is defined as

$$D_t^{\alpha} f(t) = \frac{d}{dt} J^{1-\alpha} f(t)$$

Definition3. The Caputo fractional derivative is defined as

$$D_t^{\alpha} f(t) = J^{1-\alpha} \frac{d}{dt} f(t)$$

When m and n are integers such that $m > n$, then, n^{th} - order derivative of t^n(using Euler's Gamma Function) is

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

Recently, fractional calculus was introduced to the stability analysis of nonlinear systems. The following lemmas are useful in the discussion of dynamical properties of the fractional order predator - prey system.

Lemma 4. [1] *The following linear commensurate fractional - order autonomous system*

$$D^{\alpha} x = Ax, \quad x(0) = x_0$$

is asymptotically stable if and only if $|arg\lambda| > \alpha\frac{\pi}{2}$ *is satisfied for all eigenvalues* (λ) *of matrix A. Also, this system is stable if and only if* $|arg\lambda| > \alpha\frac{\pi}{2}$ *is satisfied for all eigenvalues* (λ) *of matrix A, and those critical eigenvalues which satisfy* $|arg\lambda| = \alpha\frac{\pi}{2}$ *have geometric multiplicity one, where* $0< \alpha < 1$, $x \in R^n$ *and* $A \in R^{n\times n}$.

Lemma 5.[1] *Consider the following autonomous system for internal stability definition*

$D^{\alpha}x(t) = Ax(t), \quad x(0) = x_0$

with $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ *and its n-dimensional representation:*

$$D^{\alpha_1}x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t)$$
$$D^{\alpha_2}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \dots + a_{2n}x_n(t) \quad (1)$$
$$\dots \dots$$
$$D^{\alpha_n}x_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \dots + a_{nn}x_n(t)$$

where all $\alpha_i{}'s$ *are rational numbers between 0 and 2. Assume* m *to be the LCM of the denominator* $u_i{}'s$ *of* $\alpha_i{}'s$, *where* $\alpha_i = \frac{u_i}{v_i}$, $u_i, v_i \in Z^+$ *for* $i = 1, 2, \dots, n$ *and we set* $\gamma = \frac{1}{m}$.

Define:

$$det\begin{bmatrix} \lambda^{m\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{m\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{m\alpha_n} - a_{nn} \end{bmatrix} = 0 \qquad (2)$$

The characteristic equation (4) can be transformed to integer - order polynomial equation if all α_i*'s are rational number. Then the zero solution of system (3) is globally asymptotically stable if all roots* λ_i*'s of the characteristic (polynomial) equation (4) satisfy:*

$$|arg\ (\lambda_i)| > \gamma\frac{\Pi}{2}\forall i.$$

III. MODEL DESCRIPTION

The dynamic relationship between predator and prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology. It is well known the Lotka-Voltera prey-predator model is one of the fundamental population models, a predator -prey interaction has been described firstly by two pioneers Lotka and Voltera in two independent works [6]. Recently great considerations have been made to the fractional order models in different area of researches. The most essential property of these models is their non local property which does not exist in the integer order differential operators. We mean by this property that the next state of a model depends not only upon its current state but also upon all of its historical states. Several authors formulated fractional order systems and analyzed the dynamical and qualitative behavior of the systems [3, 4, 8, 9, 10]. In this paper, we introduce the following fractional order system of prey-predator interactions. The model for our investigation is as follows:

$$D^{\alpha_1}x(t) = rx(t)[1 - x(t)] - \frac{ax(t)y(t)}{a + x(t)}$$
$$D^{\alpha_2}y(t) = \frac{bx(t)y(t)}{a + x(t)} - cy(t) \qquad (3)$$

where the parameters r, a, b, c are positive and α_1, α_2 are fractional orders.

IV. EQUILIBRIUM POINTS, STABILITY AND NUMERICAL SOLUTIONS

Numerical solution of the fractional - order Prey - Predator system is given as follows [1]:

$$x(t_k) = \left(rx(t_{k-1})[1 - x(t_{k-1})] - ax(t_{k-1})y(t_{k-1})\right)h^{\alpha_1} - \sum_{j=v}^{k} c_j^{(\alpha_1)}x(t_{k-j})$$

$$y(t_{k-1}) = \left(bx(t_{k-1})y(t_{k-1}) - cy(t_{k-1})\right)h^{\alpha_2} - \sum_{j=v}^{k} c_j^{(\alpha_2)}y(t_{k-j})$$

where T_{sim} is the simulation time, $k = 1,2,3,\dots,N,$ for $N = [T_{sim}/h]$, and $(x(0), y(0))$ is the initial conditions. To evaluate the equilibrium points, we consider

$D^{\alpha_1}x(t) = 0$

$D^{\alpha_2}y(t) = 0$

The fractional order system has three equilibria$E_0 = (0,0)$ (trivial), $E_1 = (1,0)$ (axial) and$E_2 = \left(\frac{ac}{b-c}, br\left[\frac{1}{b-c} - \frac{ac}{(b-c)^2}\right]\right)$. The equilibrium point$E_2$is interior which corresponds to the existence of both prey and predator species provided$\frac{b}{a+1} > c$.

The Jacobian matrix of the system (3) for equilibrium$E^* = (x^*, y^*)$ is

$$J(x,y) = \begin{bmatrix} r(1-2x) - \dfrac{a^2 y}{(a+x)^2} & -\dfrac{ax}{a+x} \\ \dfrac{aby}{(a+x)^2} & \dfrac{bx}{a+x} - c \end{bmatrix} \quad (4)$$

From (4), Jacobian matrix for E_0 is

$J(E_0) = \begin{bmatrix} r & 0 \\ 0 & -c \end{bmatrix}$

and the eigenvalues of matrix $J(E_0)$ are $\lambda_1 = r$ and $\lambda_2 = -c$.Using lemma (5), the characteristic equation of the linearized system (3) at the equilibrium point E_0 is

$(\lambda^{99} - r)(\lambda^{99} + c) = 0$

Jacobian matrix for E_1 is

$$J(E_1) = \begin{bmatrix} -r & -\dfrac{a}{a+1} \\ 0 & \dfrac{b}{a+1} - c \end{bmatrix}$$

The eigen values of matrix $J(E_1)$ are $\lambda_1 = -r$ and $\lambda_2 = \frac{b}{a+1} - c$.

Most of the fractional order differential equations do not have exact analytic solutions. Hence we seek numerical techniques to analyze the behavior of the system (3). In the following examples, we illustrate the stability properties of the model by providing time plots and phase portraits.

Example 1.*Let us consider the parameter values r = 0.2; a = 1; b = 2; c = 1.3 and the derivative order*$\alpha_1 = \alpha_2 = 0.99$. *For these parameter the corresponding eigenvalues are* $\lambda_1 = -0.2$ *and* $\lambda_2 = -0.3$*for* E_1, *which satisfy conditions* $|arg\lambda| > \alpha\frac{\pi}{2}$. *It means the system (1) is stable, Fig-1. Also using lemma (5) the characteristic equation of the linearized system (3) at the equilibrium point* E_1*is*

$(\lambda^{99} - 0.2)(\lambda^{99} + 0.3) = 0$

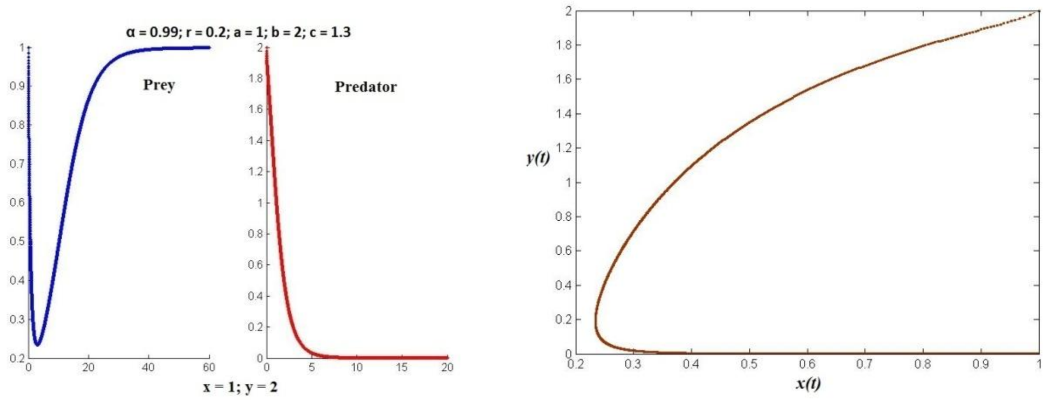

Fig1. Time plot and Phase diagram of Equilibrium point E_1

Jacobian matrix for E_2 is

$$J(E_2) = \begin{bmatrix} cr\left(1 - \frac{a(b+c)}{b(b-c)}\right) & -\frac{ac}{b} \\ \frac{r(b-c)}{a} - cr & 0 \end{bmatrix}$$

Here Trace of $J = cr\left(1 - \frac{a(b+c)}{b(b-c)}\right)$ and Det of $J = \frac{cr}{b}[b - c(a+1)]$.

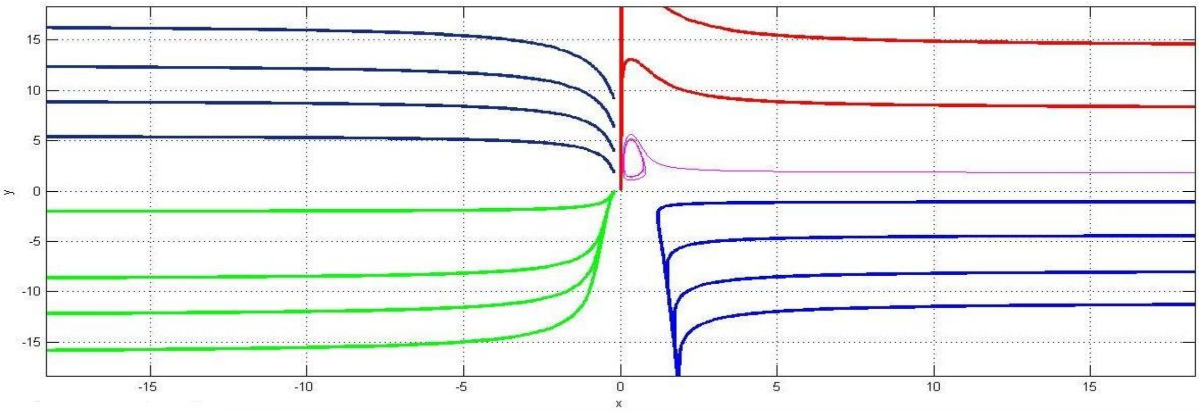

Fig 2. Phase diagram of Prey Predator Model of system (3)

Example 2.*Let us consider the parameters with values r = 1.99; a = 0.2; b = 2; c = 1.25and the derivative order* $\alpha_1 = \alpha_2 = 0.9$. *For these parameter the corresponding eigenvalues are* $\lambda_{1,2} = 0.7048 \pm i0.3537$ *for* E_2, *which satisfy conditions* $|arg\lambda| > \alpha\frac{\pi}{2}$. *It means the system (3) is stable, see Fig-3. Also using lemma (5) the characteristic equation of the linearized system (3) at the equilibrium point* E_2 *is*

$\lambda^{180} - 1.4096\lambda^{90} + 0.6219 = 0.$

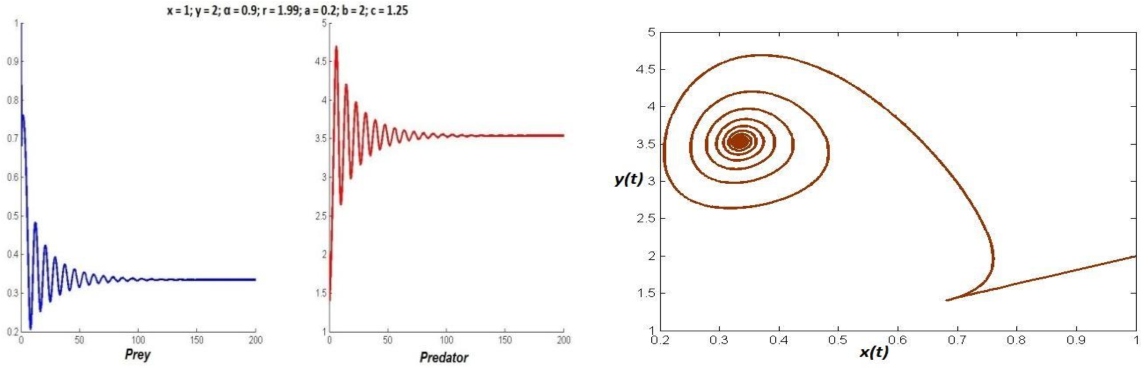

Fig 3. Time plot and Phase diagram of Equilibrium point E_2 with Stability

Example 3.*Let us consider the parameters with values r = 1.99; a = 0.1; b = 2; c = 1.5 and the derivative order* $\alpha_1 = \alpha_2 = 0.99$. *For these parameter the corresponding eigenvalues are* $\lambda_1 = 1.6172$ *and* $\lambda_2 = 0.3230$ *for* E_2, *which not satisfy conditions* $|arg\lambda| > \alpha\frac{\pi}{2}$. *It means the system (3) is unstable, see fig-4. Also using lemma (5) the characteristic equation of the linearized system (3) at the equilibrium point* E_2 *is*

$$\lambda^{198} - 1.9402\lambda^{99} + 0.5224 = 0.$$

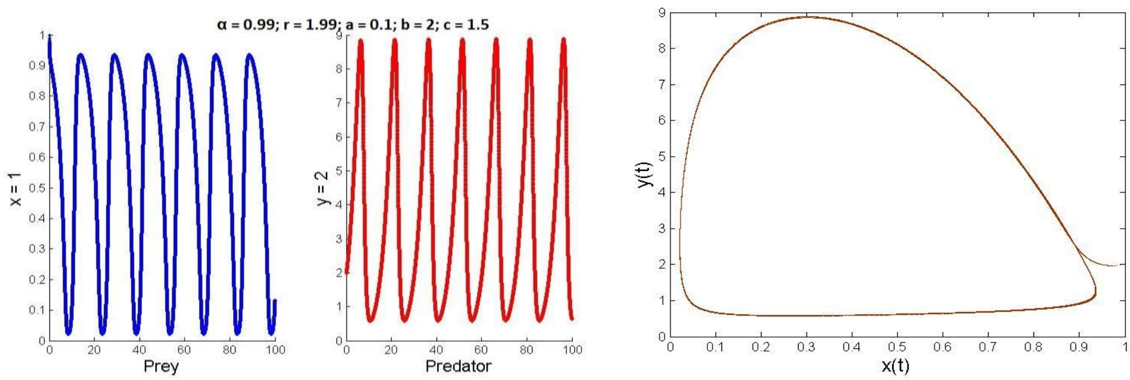

Fig4. Time plot and Phase diagram of Equilibrium point E_2 with Unstability

Example 4.*Let us consider the parameters with values r = 1.99; a = 0.17; b = 2; c = 1.3 and the derivative order* $\alpha_1 = \alpha_2 = 0.9$. *For these parameter the corresponding eigenvalues are* $\lambda_{1,2} = 0.1789 \pm i0.7665$ *for* E_2, *which satisfy conditions* $|arg\lambda| > \alpha\frac{\pi}{2}$. *It means the system (3) is Unstable, see fig-5. Also using lemma (5) the characteristic equation of the linearized system (3) at the equilibrium point* E_2 *is*

$$\lambda^{180} - 0.3579\lambda^{90} + 0.6169 = 0.$$

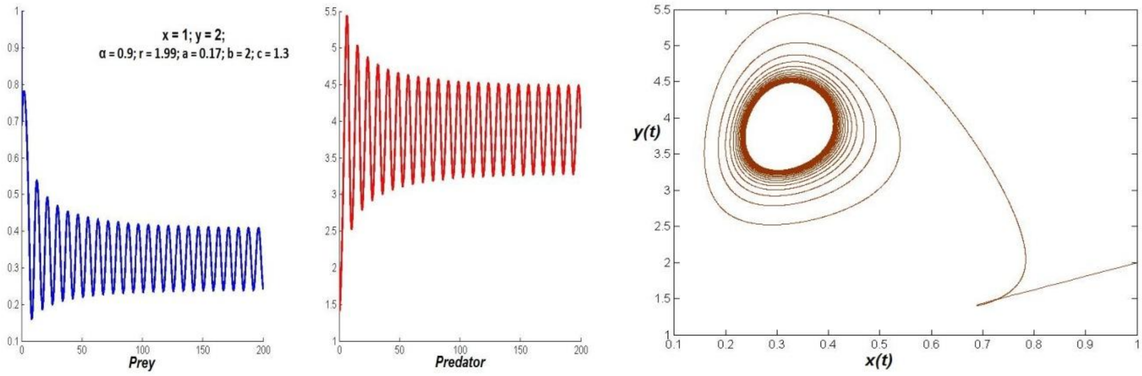

Fig 5. Time plot and Phase diagram of Equilibrium point E_2 with Limit Cycle

V. CONCLUSION

In this paper, we considered and investigated the fractional-order predator-prey model. The stability of equilibrium points is studied. Also we provided numerical simulations exhibiting dynamical behavior and stability around equilibria of the system.

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