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Stable, bounded and periodic solutions in a non-linear second order ordinary differential equation.

Eze, Everestus Obinwanne, Ukeje, Emelike and Hilary Mbadiwe Ogbu
Department of Mathematics, Michael Okpara University of Agriculture Umudike, Umuahia, Abia State.

ABSTRACT: *Results are available for boundedness and periodicity of solution for a second order non-linear ordinary differential equation. However the issue of stability of solutions in combination with boundedness and periodicity is rare in literature. In this paper, stability boundedness and periodicity of solutions have been shown to exist in a non-linear second order ordinary differential equation. This task has been achieved through the following:*

a. *The use of Lyapunov functions $v: \mathbb{R}^2 \to \mathbb{R}$ with some peculiar properties to achieve stability and boundedness in the non-linear second order ordinary differential equation.*
b. *The use of Leray-Shauder fixed point technique and an integrated equation as the mode for estimating the apriori bounds in achieving stability and boundedness of solutions.*

KEYWORDS: *Lyapunov functions, integrated equations, apriori bounds, fixed point technique, completely continous.*

I. INTRODUCTION

Consider the non-linear second order ordinary differential equation

$$\ddot{x} + a\dot{x} + h(x) = p(x) \qquad (1.1)$$

Subject to the boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi), r = 0,1 \qquad (1.2)$$

where $a > 0$ *and* $h(x), p(x)$ are continous functions depending on their argument. For the constant coefficient equation.

$$\ddot{x} + a\dot{x} + bx = p(x) \qquad (1.3$$

Ezeilo (1986)has shown that if the Ruth-Hurwitz's conditions

$$a > 0, b > 0 \qquad (1.4)$$

hold, the roots of the ordinary equation

$$\lambda^2 + a\lambda + b = 0 \qquad (1.5)$$

Have negative real parts, then asymptotic stability and ultimate boundedness of solutions can be verified for (11.3) when $p(t) = 0$. The existence of periodic solutions can be verified (1.3) when (1,4) holds: Ezeilo (1960), Tejumola (2006), Coddington and Levinson (1965), Ogbu (2006) , Ezeilo and Ogbu (2009).

A close look at equations (1.1) and (1.3) give some clue to the theorem stated below.

Theorem 1.1

Suppose there exists $a > 0, b > o$ *and* $\beta > o$ *such that*

I. $h(x) < b, \beta^2 = b,\ 1 = \frac{d}{dt}$.
II. $|h(x) - x| > 0$ for all x
III. $|h(x)| \to \infty\ |x| \to \infty$
IV. $x^2 + y^2 \to \infty$ as $|x| \to \infty, |y| \to \infty$

Then equations (1.1)- (1.2) has stable, bounded and periodic solutions when $p(t) = 0$.

Theorem 1.2

Suppose further in theorem 1.1, the condition (i) is replaced by

(i') $h^{'(x)} < b, \beta^2 \neq b,\ |a\dot{x} - p(t)| > 0$.

Then equation (1.1)- (1.2) has stable, bounded and periodic solutions when $p(t) \neq 0$.

II. PRELIMINARIES

Consider the scalar equation

$$\dot{x} = f(x), x\epsilon\mathbb{R}^n, f(0) = 0 \qquad (2.1)$$

Where f is sufficiently smooth.

Theorem 2.1

Assume that

i. $f\epsilon c'$
ii. There exists a c' function $v: \mathbb{R}^n \to \mathbb{R}$ such that $v(x) > 0\ \forall x$ and $v(x) = 0\ if\ x = 0$
iii. Along the solution paths of equation (2.1) $\dot{v} \leq 0$

Then the solution $x = 0$ of equation (2.1) is stable in the sense of Lyapunov.

Theorem 2.2

Assume that

i. $f\epsilon c'$
ii. There exists a c' function $v: \mathbb{R}^n \to \mathbb{R}$ such that $v(x) > 0\ \forall x$ and $v(x) = 0\ if\ x = 0$
iii. Along the solution paths of equation (2.1) $\dot{V} < 0,\ x \neq 0$ and $\dot{V} = 0,\ x = 0,\ i.e\ \dot{V}$ is negative definite.

Then the solution $x = 0$ of equation (2.1) is asymptotically stable in the sense of Lyapunov.

Theorem 2.3 (Yoshizawa)

Consider the system

$$\dot{x} = f(t, x, y), \dot{y} = g(t, x, y) \qquad (2.2)$$

Where f and g satisfy conditions for existence of solutions for any given initial values. Suppose there exists a function $v: \mathbb{R}^n \to \mathbb{R}$ with the first partial derivatives in its argument such that

$$v(x, y) \to \infty\ as\ x^2 + y^2 \to \infty \qquad (2.3)$$

and such that for any solution $(x(t), y(t))$ of equation (2.2)

$$\dot{V} = \frac{d}{dt}V\big(x(t), y(t)\big) \leq -\delta < 0\ if\ x^2(t) + y^2(t) \geq R > 0 \qquad (2.4)$$

Where δ and R are finite constants. Then every solution $\big(x(t), y(t)\big)$ of equation (2.2) is (uniformly) ultimately bounded with bounding constants depending on R and how $v \to +\infty\ as\ x^2 + y^2 \to \infty$. The conclusion here is that there exists a a constant $D, 0 < D < \infty$ such that

$$|x(t)| \leq D, |y(t)| \leq D \qquad (2.5)$$

The proof of theorem (1) entails establishing stability, boundedness and periodicity for equation (1.1)-(1.2) when $p(t) = 0$. That is

$$\ddot{x} + a\dot{x} + h(x) = 0 \qquad (2.6)$$

Or the equivalent system

$$\dot{x} = y$$
$$\dot{y} = -ay - h(x) \qquad (2.7)$$

Consider the function $v: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$v = \frac{1}{2}y^2 + H(x) \qquad (2.8)$$

Where $H(x) = \int_0^x h(s)ds$

Clearly the V defined above is positive semi definite. The time derivative $\dot{v}$ along the solution path of (2.7) is

$$\begin{aligned}\dot{v} &= y\dot{y} + h(x)\dot{x} \\ &= y\big(-ay - h(x)\big) + h(x)y \\ &= -ay^2 - h(x)y \\ &= -ay^2\end{aligned}$$

Which is negative definite. Therefore by Lyapunov theorem the system (2.6)-(2.7) is asymptotically stable. Hence it is stable. Therefore the system (2.6)-(2.7) is stable in the sense of Lyapunov when $p(t) = 0$

Now for the proof of boundedness in equation (2.6)-(2.7), consider the c' function $v: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 \qquad (2.9)$$

The V is defined in equation (3.4) is positive semi definite. The time derivative V along the solution path of (3.2) is

$$\begin{aligned}\dot{v} &= x\dot{x} + y\dot{y} \\ &= xy - ay^2 - yh(x) \\ &= -ay^2 - y(h(x) - x)\end{aligned}$$

Since $|h(x) - x| > 0$ for all x (condition (ii) in theorem1) then

$$\dot{v}=-ay^2 - y|h(x) - x| < 0 \qquad (2.10)$$

Without loss of generality, v is such that $\dot{v} \leq -1$ since $x^2 + y^2 \to \infty$ as $|x| \to \infty$ as $|y| \to \infty$

By Yoshizawa's theorem, equation (3.1) has a bounded solution. Therefore equation (1.1) has bounded solution when $p(t) = 0$

Now the condition (i) in theorem (1) which is $\beta^2 = b$ implies that $i\beta$ is a root of the auxiliary equation. Therefore the solution to (2.6) is the form $A \cos \beta(t) + B \sin \beta(t)$. this clearly shows that the solution is periodic. Therefore equation (2.6) is stable, bounded and periodic. Hence, the proof of theorem 1.

The proof of theorem 2 is as follows

Consider equation (1.1) or its equivalent system,

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay^2 - h(x) + p(t) \end{aligned}\right\} \qquad (2.11)$$

And the function $v: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$v = \tfrac{1}{2}y^2 + h(x) \qquad (2.12)$$

Where $H(x) = \int_0^x h(s)ds$

clearly the V defined above in equation (2.12) is positive semi-definite. The time derivative $\dot{v}$ along the solution paths of (2.11) is

$$\begin{aligned} \dot{v} &= y\dot{y} + h(x)y \\ &= y\big(-ay - h(x) + p(t)\big) + h(x)y \\ &= -ay^2 - yh(x) + yp(t) + h(x)y \\ &= -ay^2 - yp(x) \\ &= -y(ay - p(t)) \\ &= -y(a\dot{x} - p(t)) < 0 \text{ for } |a\dot{x} - p(t)| \end{aligned}$$

This is a negative definite.

Therefore by Lyapunov theorem, the system (2.11) is asymptotically stable in the sense of Lyapunov.

Next we proceed to establish boundedness and periodicity in equation (1.1) a parameter λ, dependent equation.

$\ddot{x} + a\dot{x} + h_\lambda(x) = \lambda p(t)$ (2.13)

where $h_\lambda(x) = (1 - \lambda)bx + \lambda h(x)$ (2.14)

where λ is in the range of $0 \leq \lambda \leq 1$ and b is a constant satisfying (1.4). The equation (2.13) is equivalent to the system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay^2 - h(x) + p(t) \end{aligned}\right\} \qquad (2.15)$$

the system of equation (2.10) can be represented in the vector form

$\dot{x} = Ax + \lambda F(t, x).$ (2.16)

where

$$x = \begin{Bmatrix} x \\ y \end{Bmatrix}, A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, F = \begin{bmatrix} 0 \\ p(t) - h(t) + bx \end{bmatrix} \qquad (2.17)$$

We remark that the equation (3.8) reduces to a linear equation

$\ddot{x} + a\dot{x} + bx = 0$ (2.18)

when $\lambda = 0$ and to equation (1.1) when $\lambda = 1$.

If the roots of the auxiliary equation (2.18) has no root of the form

$\beta^2 \neq b, \beta^2 \neq 0$ (2.19)

(β is an integer), then equation (1.1) –(1.2) has at least one 2π periodic solution that is the matrix. As defined in equation (2.17) has no imaginary roots so that the matrix $(e^{-2\pi A} - 1)$ where (1) is the identity (2×2) matrix is invertible. Therefore x is a 2π periodic solution of equation (3.11) if and only if $x = \lambda TX, 0 \leq \lambda \leq 1$ (2.20)

Where the transformation T is defined by

$$(TX)(t) = \int_0^{2\pi} \left(e^{(-2\pi A)} - 1\right)^{-1} e^{((t-s)A)} F\big(S, X(s)\big)ds \qquad (2.21)$$

Hale (1963)

Let s be the space of all real valued continuous 2-vector function $x(t) = (\bar{x}(t), \bar{y}(t))$ which are of period 2π. If the mapping T is completely continuous mapping of S into itself. Then existence of a 2π periodic solution to equation (1.1)-(1.2) correspond to $X\epsilon S$ satisfying equation (2.20) for $\lambda = 1$. Finally using Scheafer's lemma (Scheafer 1955) We establish that

$|x|_\infty \leq c_6, |\dot{x}|_\infty \leq c_3$ (2.22)

where the c's are the apriori bounds.

III.RESULTS

Let $x(t)$ be a possible 2π periodic solution of equation (2.13). The main tool to be used here in this verification is the function $W(x, y)$ defined by

$W(x,y) = \frac{1}{2}y^2 + H_\lambda(x)$ (3.1)

where $H_\lambda(x) = \int_0^x h_\lambda(s)ds$

The time derivative $\dot{W}$ of equation (3.1) along the solution paths of (2.15) is

$$\begin{aligned} \dot{w} &= y\dot{y} + h_\lambda(x)\dot{x} \\ &= -ay^2 - h_\lambda(x)y + \lambda p(t)y + h_\lambda(x)y \\ &= -ay^2 + \lambda p(t)y \qquad (3.2) \end{aligned}$$

Integrating the equation (3.2) with respect to t from $t = 0$ to $t = 2\pi$

$$\int_0^{2\pi} \dot{w}dt = \int_0^{2\pi} -a\dot{x}^2dt + \int_0^{2\pi} \lambda p(t)\dot{x}\ dt$$

$$[W(t)]_0^{2\pi} = \int_0^{2\pi} -a\dot{x}^2dt + \lambda\int_0^{2\pi} p(t)\dot{x}\ dt$$

Since $W(0) = W(2\pi)$ implies that $[W(t)]_0^{2\pi} = 0$ because of 2π periodicity. Thus

$$0 \le -\int_0^{2\pi} a\dot{x}^2dt + \lambda\int_0^{2\pi} p(t)\dot{x}\ dt$$

$$a\int_0^{2\pi} \dot{x}^2dt \le |\lambda|\,|p(t)|\int_0^{2\pi} \dot{x}\ dt \qquad (3.3)$$

Since $|\lambda| \le 1$ and $p(t)$ is continous then

$$\int_0^{2\pi} a\dot{x}^2dt \le c_1(2\pi)^{1/2}\left(\int_0^{2\pi} \dot{x}dt\right)^{1/2}$$

by Schwartz's inequality. Therefore

$$\left(\int_0^{2\pi} \dot{x}^2dt\right)^{1/2} \le c_1 2\pi^{1/2} \equiv c_2$$

That is $\left(\int_0^{2\pi} \dot{x}^2dt\right)^{1/2} \le c_2$

Now since $x(0) = x(2\pi)$, it is clear that there exists $\dot{x}(\tau) = 0$ for $\tau \in [0, 2\pi]$. Thus the identity

$$\dot{x}(t) = \dot{x}(\tau) + \int_0^{2\pi} \ddot{x}^2\,ds$$

$$= \int_0^{2\pi} \ddot{x}^2(s)\,ds$$

$max_{0\le t\le 2\pi}|\dot{x}(t)|dt \le \int_0^{2\pi}|\ddot{x}(t)|dt \le 2\pi^{1/2}\left(\int_0^{2\pi}\ddot{x}^2(s)ds\right)^{1/2}$ (3.4)

By Schwartz's inequality.

Using Fourier expansion of

$$x \tilde{} \sum_{r=0}^{\infty}(ar\cos 2\pi l + br\sin 2\pi l)$$

and its derivative in Ezeilo and Onyia (1984) we obtain

$$\int_0^{2\pi} \ddot{x}^2dt \le |\lambda||p(t)|\int_0^{2\pi} \dot{x}dt$$

$\int_0^{2\pi}\ddot{x}^2dt \le c_1(2\pi)^{1/2}\left(\int_0^{2\pi}\ddot{x}^2dt\right)^{1/2}$ by Schwartz's inequality. Therefore

$\left(\int_0^{2\pi}\ddot{x}^2dt\right)^{1/2} \le c_1(2\pi)^{1/2} \equiv c_2$ (4.5)

From equation (4.5)

$$max_{0\le t\le 2\pi}|\dot{x}(t)| \le (2\pi)^{1/2}.c_1(2\pi)^{1/2} \equiv c_3$$

$|\dot{x}|_\infty \le c_3$ (4.6)

Now integrating equation (3.8) with respect to t from $t = 0$ to $t = 2\pi$, we obtain

$\int_0^{2\pi}\ddot{x}dt + \int_0^{2\pi}a\dot{x}dt + \int_0^{2\pi}h_\lambda(x)dt = \int_0^{2\pi}\lambda p(t)dt.$ (4.7)

Using equation (3.9) on (4.8) we obtain

$$\int_0^{2\pi}\ddot{x}dt + \int_0^{2\pi}a\dot{x}dt + \int_0^{2\pi}(1-\lambda)bx\,dt + \int_0^{2\pi}h(x)dt = \int_0^{2\pi}\lambda p(t)dt.$$

using the 2π periodicity of solution in equation (1.2) then equation (4.8) yields

$\int_0^{2\pi}(1-\lambda)bx\,dt + \int_0^{2\pi} h(x)dt = \int_0^{2\pi} \lambda p(t)dt.$ (4.8)
The continuity of $p(t)$ assures boundedness and the fact that $0 \le \lambda \le 1$, the right hand side of equation (4.8) is bounded. That is
$\left|\int_0^{2\pi} \lambda p(t)dt.\right| \le c_4$ (4.9)
So
$\left|\int_0^{2\pi}(1-\lambda)bxdt + \int_0^{2\pi} \lambda h(x)dt\right| \le c_4$ (4.10)
Therefore given $\alpha > 0$, there exists $\eta > 0$ such that $\tau \in [0, 2\pi]$
$|x(\tau)| \le c_5$ (4.11)
$\tau = 0$ We are done.
Suppose NOT, i.e $x(\tau) \ne 0$ for any τ then equation (4.9) yields
$\int_0^{2\pi}(1-\lambda)b|x|dt + \int_0^{2\pi}|\lambda||h(x)|dt > \int_0^{2\pi}(1-\lambda)b\eta dt + \int_0^{2\pi} \lambda\alpha dt > 2\pi(1-\lambda)b\eta + 2\pi\lambda a$ (4.12)
but equation (4.13) implies that $\int_0^{2\pi}(1-\lambda)bxdt + \int_0^{2\pi} \lambda h(x)dt$ is no more bounded.
This is a negation of equation (4.10). Thus $|h(x)| \to \infty$ as $|x| \to \infty$ and equation (4.12) holds. The identity.
$x(t) = x(\tau) + \int_\tau^t \dot{x}\, dt$ Holds.
Thus

$$max_{0\le t\le 2\pi}|x(t)| \le |x(t)| + \int_0^{2\pi} |\dot{x}(t)|dt \le c_5 + (2\pi)^{1/2}\left(\int_0^{2\pi} \dot{x}^2 dt\right)^{1/2}$$

By Schwartz's inequality
$\le c_5 + (2\pi)^{1/2}.c_2$ by equation (4.4)
Thus $|x|_\infty \le c_5 + (2\pi)^{1/2}.c_2 \equiv c_6$
so
$|x|_\infty \le c_6$ (4.13)

IV. CONCLUSION:-

By equations (4.7) and (4.13) equation (2.22) is established and hence complete proof of theorem 2. Note also that equation (4.7) and (4.12) indicate that the solution $x(t)$ and $\dot{x}(t)$ are bounded.

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NOTE

We are publishing this article in honour of late Professor Hillary Mbadiwe Ogbu who died on the 2nd of June 2014 and May His gentle soul rest in Peace. Amen.

This was his last Article --- THE FINAL EQUATION