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Inventory Model (Q, R) With Period of Grace, Quadratic Backorder Cost and Continuous Lead Time

Dr. Martin Osawaru Omorodion

Joseph Ayo Babalola University, Ikeji-Arakeji, Nigeria.

ABSTRACT: *The paper considers the simple economic order model where the period of grace is operating, the lead time is continuous and the backorder cost is quadratic. The lead time follows a gamma distribution. The expected backorder cost per cycle is derived and averaged over all the states of the lead time L. Next we obtain the expected on hand inventory. The Lead time is taken as a Normal variate. The expected backorder costs are, derived after which the expected on hand inventory is derived. The cost inventory costs for constant lead times is then averaged over the states of the lead times which is taken as a normal distribution.*

***KEY WORDS**: Continuous Lead Times, Gamma Distribution, Normal Distribution, Period of Grace, Bessel Functions, Inventory on hand, Expected Backorder Costs and Chi-Squared Distribution.*

Demand

The cost of a backorder $C_\beta(t)$ for backorder of length t = $C_\beta(L-z-p)$ = $b_1+b_2(L-z-p)+b_3(L-z-p)^2$ where p is the grace period, where the system is out of stock in the time interval z to z tdz. After the re-order point is reached the R+Y was demanded in the time z and a demand occurred in time d~~z~~. The longer the period of grace the greater the reduction in inventory costs.

SIMPLE INVENTORY MODEL WITH PERIOD OF GRACE, QUADRATIC BACKORDER COST AND CONTINUOUS LEAD TIME

I. INTRODUCTION

In this inventory model there is a period of grace before backorder costs are incurred. In Hadley (1972), the general model was stated. This paper derives the backorder costs when the backorder cost is a quadratic cost depending upon the length of time of backorder after the grace period. The various costs such as expected backorder costs and expected on hand inventory are derived, to arrive at the total costs of holding inventory. Demand over the lead time is normally distributed and lead time is a gamma variate.

II. LITERATURE REVIEW

Saidey (2001), in his paper Inventory Model with composed shortage considered grace period (perishable delay in payment) before setting the account with the supplier or producer.

Schemes (2012) also considered credit terms which may include an interest free grace period as much as 30 days in his paper 'Inventory models with shelf Age an Delay Dependent Inventory costs'.

Hadley and Whitin gave a simple general model for period of grace.

Zipkin (2006), treats both fixed and random lead times and examines both stationary and limiting distributions under different assumptions.

III. DERIVATION

Let the period of grace be p, where the period of grade is the period for which a backorder bears no cost. Let H(L) be the probability density function of the lead time L.

$C_\beta(t)$ is the cost of a backorder of length t.

H(L) = $\dfrac{\alpha^k L^{k-1} esp(-\alpha L)}{\Gamma(k)}$ $\alpha > 0$…………….. (1)

In the analysis k would take on integer values only.
The p.d.f. of demand over the lead time L.

$$g\left(\frac{x-DL}{\sqrt{\sigma 2L}}\right) = \frac{esp\left[-\frac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2\right]}{\sqrt{2\pi\sigma^2 L}} \quad -\infty > x < \infty \text{ ………………… (2)}$$

Let R + Y, 0 <Y<Q be the inventory position at time O, then if the system is out of stock in the time internal z to z + dz after the re-order point is reached then R +Y was demanded in the time Z and a demand occurred in time dz.

Length of time of backorder = L-z length of time of backorder which bears a cost
= L – z – p.
Cost of a backorder = C_β (L – z – p)

Where the C_β (t) is the cost of a backorder = $b_1 + b_2 (L - z - p) + b_3 (L - z - p)^2$. (3)

Expected backorder cost per cycle G_1 (Q, R).

$$= D\int_0^\infty \int_0^Q \int_0^{L-P} \frac{C_\beta(L-z-p)}{\sqrt{\sigma^2 L}}.H(L).g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 L}}\right).dzdydL \text{ …………….. (4)}$$

Let v = z + p

$$G_1 (Q,R) = D\int_0^\infty \int_0^Q \int_0^L \frac{C_\beta(L-v)}{\sqrt{\sigma^2 L}} H(L) g\left(\frac{R+Y+Dp-Dv}{\sqrt{\sigma^2 L}}\right) dvdYdL \text{ ……………...(5)}$$

Making use of equation (2)

$$G_1(Q,R) = D\int_0^\infty \int_0^Q \int_0^L \frac{C_\beta(L-v)}{\sqrt{2\pi\sigma^2 L}}.H(L) esp\frac{1}{2}\left(\frac{R+Y-Dp-Dv}{\sqrt{\sigma^2 L}}\right) dvdYdL \text{ …. (6)}$$

Substituting for C_β (L-v) from 3

$$G_1(Q,R) = D\int_0^\alpha \int_0^Q \int_0^L (b_1+b_2(L-v) + b_3 (L-v)^2 + H(L) \text{ ………………..(7)}$$

$$esp\ -1/2 \left(\frac{R+Y+DP-DV)}{\sqrt{\sigma^2 L}}\right. dvdYdL$$

Noting that

$$D\int_0^L \frac{(b_1+b_2(L-v)+b_3(L-v)^2}{\sqrt{\sigma^s L}} esp - 1/2\left(\frac{R+Y-Dp-Dv}{\sqrt{\sigma^2 L}}\right)^2 dv \text{ ……………. (8)}$$

=

$$\left(\frac{b_1+b_3\sigma^2L}{D^2}+\frac{b_3\sigma^2L}{D^2}\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2L}}\right)^2-\frac{\sqrt{\sigma^2L}b_2}{D}\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2L}}\right)F\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2L}}\right)\right.$$

$$-\left(\frac{\sqrt{\sigma^2}Lb_2}{D}-\frac{b_3\sigma^2L}{D^2}\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2L}}\right)\right)g\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2L}}\right) \quad \text{.....................} (9)$$

Substituting into $G_1(Q,R)$ of (7) and changing the range of Y

$$=\int_0^\infty H(L)\int_{R+Dp}^{R+Q+Dp}\left(\frac{b_1+b_3\sigma^2L}{D^2}+\frac{b_3\sigma^2L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)^2-\frac{b_2\sqrt{\sigma^2}L}{D}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)\right)F\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dxdL$$

$$=-\int_0^\infty H(L)\int_{R+Dp}^{R+Q+Dp}\left(\sqrt{\sigma^2Lb_2}+b_3\frac{\sqrt{\sigma^2L}}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)\right)g\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dxdL \quad \text{..........} (10)$$

Let $G_1(Q,R) = \int_{R+Dp}^{R+Q+Dp} G_2(x)dx$ .... (11)

Hence from equation (10)

$$G_2(x) = -\int_0^\infty H(L)\left(\frac{b_1+b_3\sigma^2L}{D^2}+b_3\frac{\sigma^2L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)\frac{b_2}{D}(x-DL)F\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)\right)dL$$

$$-\frac{b_2}{D}(x-DL)F\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL$$

$$-\int_0^\infty H(L)\left(\frac{b_0\sigma L^{1/2}}{D}-b_3\frac{\sigma^2L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)\right)g\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL \quad \text{..................} (12)$$

Substituting for H (L) from 1 and simplifying

$$G_2(x) = \frac{\alpha^k}{\overline{)(K)}}\int_0^\infty\left(b_1L^{K-1}+b_3\frac{\sigma^2L^k}{D^2}+\frac{b_3x^2L^{k-1}}{D^2}\right.$$

$$\left.-2\frac{b_3xL^K}{D}+\frac{b_3L^{k+1}}{D^2}-\frac{b_2xL^{k-1}}{D}+b_2L^k\right)\text{esp}\,(-\alpha L)\text{ F}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL$$

$$-\frac{\alpha^k}{\overline{)k}}\int_0^\infty\left(\frac{\alpha L^{k-1/2}b_2}{D}-b_3\frac{\sigma L^{k-1/2}}{D^2}x+b_3\frac{\sigma L^{k+1/2}}{D}\right)\text{esp}\,(\alpha\text{ L})\text{g}\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL \quad \text{.......} (13)$$

Re-arranging terms

$$G_2(x) = \frac{\alpha^k}{\overline{)k}}\left[L^{k-1}\left(b_1 + \frac{b_3 x^2}{D^2} - \frac{b_2 x}{D} \right)\right]$$

$$-L^k\left(\frac{b_3\sigma^2 -}{D^2} - 2\frac{b_3 x}{D} + b_2 \right) + \frac{b_3 L^{k-1}}{D^2} \Bigg) esp(-\alpha L)$$

$$F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)dL - \frac{\alpha^k\sigma}{\overline{)k}}\left[L^{k-1/2}\left(\frac{b_2}{D} - \frac{b_3 x}{D^2}\right)\right]$$

$$+\frac{b_3 L^{k+1/2}}{D}\Bigg] esp(-\alpha L) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)dL \quad \ldots\ldots (14)$$

Integrating

$$\int_0^\infty \frac{1}{\sqrt{\sigma^2 L}} H(L) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)dL = \int_0^\infty esp\,\frac{(-\alpha L)L^{k-1}\alpha^k}{\sqrt{\sigma^2 L}.\overline{)k}} g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)dL$$

$$= \frac{\alpha^k}{\sqrt{2\pi}\overline{)k}}\int_0^\infty L^{k-3/2} esp\left(\frac{Dx}{\sigma^2}\right) esp\left(\frac{-x^2}{2\sigma^2 L} - L\left(\frac{2\alpha\sigma^2 + D^2}{2\sigma^2}\right)\right)$$

$$= \frac{\alpha^k}{\sigma\sqrt{2\pi}} esp\frac{\left(\frac{Dx}{\sigma^2}\right)}{\overline{)k}}\left[2\left(\frac{x^2}{2\alpha\sigma^2 + D^2}\right)^{1/2(K-1/2)} K_{k-1/2}\left[\frac{x}{\sigma^2}\left(2\alpha\sigma^2 + D^2\right)^{1/2}\right]\right]$$

If k is an integer then

$$K_{k-1/2}(z) = k_{1/2}(z)\sum_{j=0}^{k-1}\frac{(k+j-1)}{j!(k-j-1)!}(2z)^{-j}$$

Where $K_{1/2}(z) = \frac{\sqrt{\pi}}{\sqrt{2}}(z)^{-1/2} esp(-z)$

Hence $K_{k-1/2}(z) = \sqrt{\pi}\sum_{j=0}^{k-1}\frac{(k+j-1)}{j!(k-j-1)!}(2z)^{-j-1/2} esp(-Z)$

And Letting

$$\vartheta^2 = 2\alpha\ \sigma^2 + D^2$$

$$G_2(x) = esp\left(\frac{Dx}{\sigma^2}\right)\frac{\alpha^k}{\overline{)(k)}\sqrt{2\pi}}\left[b_1 + \frac{b_3 x^2}{D^2} - \frac{b_2 x}{D}\right]$$

$$\sum_{z=1}^{k}\frac{(k-1)}{\alpha^2(k-z)!}\left(2D\left(\frac{x}{\theta}\right)^{k-Z+1/2} K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right) + 2x\left(\frac{x}{\theta}\right)^{k-z-1/2} K_{k-z-1/2}\left(\frac{x\sigma^2}{\sigma^2}\right)\right)$$

$$+\left(\frac{b_3\sigma^2}{D^2}-\frac{2b_3x}{D}+b_2\right)\sum_{z=1}^{k+1}\frac{k!}{\alpha^z(k-z+1)!}\left(2D\left(\frac{x}{\theta}\right)^{k-z+3/2}\right.$$

$$\left.K_{k-Z+3/2}\left(\frac{x\theta}{\sigma}\right)+2x\left(\frac{x}{\theta}\right)^{k-z+1/2}K_{k-Z+1/2}\left(\frac{x\sigma}{\sigma^2}\right)\right)$$

$$+\frac{b_3}{D^2}\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z(k-z+2)!}\left(2D\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z-21/2}\left(\frac{x\theta}{\sigma^2}\right)+2x\left(\frac{x}{\theta}\right)^{k-z+11/2}K_{k-z-11/2}\left(\frac{x\theta}{\sigma^2}\right)\right)$$

$$-\frac{\alpha^k esp\left(\frac{Dx}{\sigma^2}\right)}{\Gamma(k)\sqrt{2\pi}}\left[2\left(\frac{b_2}{D}-\frac{b_3x}{D^2}\right)\left(\frac{x}{\theta}\right)^{k+1/2}K_{k+1/2}\left(\frac{x\theta}{\sigma^2}\right)+\frac{2b_3}{D}\left(\frac{x}{\theta}\right)^{k+11/2}K_{k+11/2}\left(\frac{x\theta}{\sigma^2}\right)\right]$$

Re-arranging terms

$$G_2(x)=\frac{\alpha^k esp\left(\frac{Dx}{\sigma^2}\right)}{2\sigma}\frac{b_1}{\Gamma(k)}\left[\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^2(k-z)!}\left(2D\left(\frac{x}{\theta}\right)^{k-z+1/2}\right)\right]$$

$$K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right)+2x\left(\frac{x}{\theta}\right)^{k-z-1/2}K_{k-z-1/2}\left(\frac{x\vartheta}{\sigma^2}\right)$$

$$+\frac{\alpha^k esp\left(\frac{Dx}{\sigma^2}\right)}{2\sigma\sqrt{2\pi}\Gamma(k)}\frac{b_2}{D}\left[-\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\right]\left(2.D\theta\left(\frac{x}{\theta}\right)^{k-z+11/2}+K_{k-z-1/2}\left(\frac{x\theta}{\sigma}\right)\right.$$

$$\left.+2\theta\left(\frac{x}{\vartheta}\right)^{k-z+11/2}+K_{k-z-1/2}\left(\frac{x\theta}{\sigma^2}\right)\right)$$

$$+\sum_{z=1}^{k+1}\frac{k!}{\alpha^z(k-z+1)!}(2D^2\left(\frac{x}{\theta}\right)^{k-z+11/2}K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right)$$

$$+2\theta D\left(\frac{x}{\theta}\right)^{k-z+11/2}K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right)$$

$$-2\sigma^2\left(2\left(\frac{x}{\theta}\right)^{k-11/2}K_{k-11/2}\left(\frac{x\theta}{\sigma^2}\right)\right)$$

$$+\alpha^k\frac{esp\left(\frac{Dx}{\sigma^2}\right)b_3}{2\sigma\sqrt{2\pi}\Gamma(k)}\left[\sum_{z=0}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\left(\frac{2\theta^2}{D}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-2+1/2}\left(\frac{x\theta}{\sigma^2}\right)+\frac{2}{D^2}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z+1/2}\left(\frac{x\theta}{\sigma_2}\right)\right)\right]$$

$$+\sum_{z=1}^{k-11}\frac{k!}{\alpha^{z}(k-z+1)!}\left(\frac{2\sigma^{2}}{D}\left(\frac{x}{\sigma}\right)^{k-z+11/2}K_{k-2+11/2}\left(\frac{x\theta}{\sigma^{2}}\right)\right)$$

$$+\frac{2\theta\sigma^{2}}{D^{2}}\left(\frac{x}{\theta}\right)^{k-z-11/2}K_{k-Z+1/2}\left(\frac{x\theta}{\sigma^{2}}\right)$$

$$-\sum_{z=1}^{k+1}\frac{k!}{\alpha^{z}(k-z+1)!}\left(\frac{4}{D^{2}}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z+1/2}\left(\frac{x\theta}{\sigma^{2}}\right)\right.$$

$$+\frac{4\theta^{2}}{D}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z+1/2}\left(\frac{x\theta}{\sigma^{2}}\right)$$

$$+\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^{z}(k-z+2)!}\left(\frac{2}{D}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z+21/2}\left(\frac{x\theta}{\sigma^{2}}\right)+\frac{2\theta}{D^{2}}\left(\frac{x}{\theta}\right)^{k-z+21/2}K_{k-z+11/2}\left(\frac{x\theta}{\sigma^{2}}\right)\right)$$

$$+\frac{2\alpha^{k}esp\left(\frac{Dx}{\sigma^{2}}\right)}{\Gamma(k)}\left[\frac{\sigma\theta}{D^{2}\sqrt{2\pi}}\left(\frac{x}{\theta}\right)^{k+11/2}K_{k+1/2}\left(\frac{x\theta}{\sigma_{2}}\right)-\frac{\sigma}{\sqrt{2\pi D^{2}}}\left(\frac{x}{\theta}\right)^{k+11/2}K_{k-11/2}\left(\frac{x\theta}{\sigma^{2}}\right)\right] \quad (16)$$

From equation 11

$$G_{1}(Q_{1}R)=\int_{R+Dp}^{R+Q+Dp}G_{2}(x)dx$$

Where $G_1(Q,R)$ is the expected backorder cost per cycle

Expanding

$$G_{1}(Q,R)=\int_{R+Dp}^{R+Q+Dp}G_{2}(x)dx-\int_{R+Q+Dp}^{\infty}G_{2}(x)dx \quad \ldots\ldots(17)$$

Integrating $G_2(x)$ with respect to x from z to ∞ and applying

$$\int H(L)g\left(\frac{x-DL}{\sqrt{\sigma^{2}L}}\right)dL$$

$$=\int_{0}^{\infty}\frac{1}{\sqrt{2\pi\sigma^{2}L}}esp-1/2\left(\frac{x-DL}{\sqrt{\sigma^{2}L}}\right)^{2}\frac{\alpha^{k}L^{k-1}esp(-\alpha L)}{\Gamma(k)}dL \quad \ldots\ldots (18)$$

$\int_{Q}^{\infty}G_{2}(x)dx$ and letting $\lambda=\left(\frac{\theta-D}{\sigma^{2}}\right)$

$$=\frac{2\alpha^{k}b_{1}}{4\Gamma(k)}\left[\sum_{z=1}^{k}\frac{(k-1)}{\alpha^{z}(k-z)!}\left[2D\sum_{j=0}^{k-z}\frac{(k-z+j)!}{j(k-z-j)!}\right.\right.$$

$$\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} .\chi^2\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)\frac{1}{(\theta\lambda)^{k-z+1}}(k-z-j)!$$

$$+\theta\sum_{j=0}^{k-z-1}\frac{(k-z-1-j)!}{j(k-z-1-j)!}\frac{1}{(\theta\lambda)^{k-z+1}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z-j+1}\right)$$

$$+\frac{2\alpha^k b_2}{4D\overline{)(k)}}\left[-\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\left[D\theta\sum_{j=0}^{k-z}\frac{(k-z+j)!}{j(k-z-j)!}\right.\right.$$

$$\frac{1}{(\theta\lambda)^{k-z+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)$$

$$\left.+\theta\sum_{j=0}^{k-z-1}\frac{(k-z-1+j)!}{j!(k-z-1-j)!}\frac{(k-z-j)!}{(\theta\lambda)^{k-z+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)\right]$$

$$+\sum_{z=1}^{k+1}\frac{k!}{\alpha^z(k-z+1)!}\left[D^2\sum_{j=0}^{k-z+1}\frac{(k-z+1-j)!}{j!(k-z+1-j)!}\frac{(k-z+1-j)!}{(\theta\lambda)^{k-z+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\right]$$

$$\chi^2\left(\frac{2\lambda Q}{2(k-z+2-j}\right)+2\theta D\sum_{j=0}^{k-z}\frac{(k-z+j)!}{j!(k-z-j)!}\frac{(k-z+1-j)!}{(\theta\lambda)^{k-z+2}}$$

$$\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z+2-j)}\right)\Bigg]$$

$$-2\sigma^2\sum_{j=0}^{k}\frac{(k+j)!}{j!(k-j)!}\frac{(k-j)!}{(\theta\lambda)^{k+1}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k+1-j)}\right)\Bigg]$$

$$+\frac{\alpha^k b_3}{2\overline{)(k)}}\left[\sum_{z=0}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\left[\frac{2\theta^2}{D}\sum_{j=0}^{k-z}\frac{(k-z+j)!}{j!(k-z-j)!}\right.\right.$$

$$\frac{(k-z+2-j)!}{(\theta\lambda)^{k-z+3}}\left(\frac{2\vartheta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z+3-j)}\right)$$

$$+\frac{2\theta^3}{D^2}\sum_{j=0}^{k-z-1}\frac{(k-z+1+j)}{j!(k-z-1-j)!}\frac{(k-z+2-j)!}{(\theta\lambda)^{k+z+3}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z+3-j)}\right)$$

$$+\sum_{z=1}^{k+1}\frac{k!}{\alpha^2(k-z+1)!}\left[\frac{2\sigma^2}{D}\sum_{j=0}^{k-z+1}\frac{(k-z+1+j)!}{j!(k+1-z-j)!}(k-z+1-j)!\left(\frac{1}{\theta\lambda}\right)^{k-z+2}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\right.$$

$$\left.\chi^2\left(\frac{2\lambda Q}{2(k-z+2-j)}\right)\right]$$

$$+\frac{2\theta\sigma^2}{D^2}\sum_{j=0}^{k-z}\frac{(k-z+j)!}{j!(k-z-j)!}(k-z+1-j)!\frac{1}{(\theta\lambda)^{k-z+2}}$$

$$\chi^2\left(\frac{2\lambda Q}{2(k-z+2-j)}\right)\Bigg]\Bigg]-\sum_{z=1}^{k-1}\frac{k!}{\alpha^z(k-z+1)!}\Bigg[\frac{4}{D^2}\sum_{j=0}^{k-z+2}$$

$$\frac{(k-z+2+j)!}{j!(k-z+2-j)!}\frac{(k-z+2-j)!}{}\frac{1}{(\theta\lambda)^{k-z+3}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}$$

$$\chi^2\left(\frac{2\lambda Q}{2(k-z-j+3)}\right)\Bigg]+\frac{4\theta^2}{D}\sum_{j=0}^{k-z+1}\frac{(k-z+1+j)!}{j!(k-z+1-j)!}$$

$$(k-z+2-j)!\left(\frac{1}{\theta\lambda}\right)^{k-z+3}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k-z-j+3)}\right)\Bigg]$$

$$+\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z(k-z+2)!}\left(\frac{2}{D}\sum_{j=0}^{k-z+2}\frac{(k-z+2+j)!}{j!(k-z+2-j)!}\right)(k-z+2-j)!$$

$$\left(\frac{1}{\theta\lambda}\right)^{k-z+3}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2Q}{2(k-z+3-j)}\right)$$

$$+\frac{2\theta}{D^2}\sum_{j=0}^{k-z+1}\frac{(k-z+2-j)!}{j!(k-z+1-j)!}\left(\frac{1}{\vartheta\lambda}\right)^{k-z+3}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2Q}{2(k-z+3-j)}\right)$$

$$-\frac{2\sigma^2}{D}\sum_{j=0}^{k+1}\frac{(k+1+j)!}{j!(k+1-j)!}\frac{(k+1-j)!}{(\vartheta\lambda)^{k+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k+2-j)}\right)$$

$$+\frac{2\sigma^2\theta}{D}\sum_{j=0}^{k}\frac{(k+j)!}{j!(k-j)!}\frac{(k+1-j)!}{(\theta\lambda)^{k+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\chi^2\left(\frac{2\lambda Q}{2(k+2-j)}\right)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Let G_3 (z) equal the b_1 factor of (19) ……………………………………………(20)
Let G_4 (z) equal the b_2 factor of (19) ……………………………………………..(21)
Let G_5 (z) equal the b_3 factor of (19) ……………………………………………..(22)

Hence from (17)
The expected backorder cost per cycle
= b_1 (G_3(R + Dp) – G_3 (R + Q+ Dp) + b_2 (G_4 (R + Dp)
- G_4 (R +Q +Dp)) + b_3 (G_5 (R + Dp) – G_5 (R + Q +Dp)) …………………..…(23)
Hence the expected cost per year

$$=\frac{b_1}{Q}\big(G_3(R+Dp)-G_3(R+Q+Dp)\big)+\frac{b_2}{Q}\big(G_4(R+Dp)-G_4(R+Q+Dp)\big)+\frac{b_3}{Q}\big(G_5(R+Dp)-G_5(R+Q+Dp)\big)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Next we obtain the expected on hand inventory at any time.

Let (D, R) be the expected on hand inventory at any time

Let G_6 (x) be the probability density function of the quantity on hand x at anytime t. x control be less or above the re-order level R. 0<x<R

$$D(Q,R) = \int_0^R xG_6(x)dx + \int_R^{R+Q} xG_6(x)dx$$

If x lies above R

Where $G_6(x) = \frac{1}{Q}\int_R^{R+Q} esp - 1/2\left(\frac{v - x - DL}{\sqrt{\sigma^2 L}}\right)^2 dx$

If x lies above R

$$G_6(x) = \frac{1}{Q}\int_0^{R+R} esp\left(-\frac{1}{2}\left(\frac{v - x - DL}{\sqrt{\sigma^2 t}}\right)dv\right.$$

Given that the system was in state v and v-x was demanded.

Which gives, expressing in k standard deviations of stock

$$G_1(x) = \frac{1}{Q}\left(F(k - \frac{x}{\sqrt{\sigma^2 L}}\right) - F\left(k + \frac{(Q - x)}{\sqrt{\sigma^2 L}}\right) \qquad 0 < x < R$$

$$G_1(x) = \frac{1}{Q}\left(1 - F\left(k + \frac{(Q - x)}{\sqrt{\sigma^2 L}}\right)\right) \qquad R < x < R + Q$$

Hence D(Q, R)= $\frac{1}{Q}\int_0^R \left(xF\left(\frac{R - x - DL}{\sqrt{\sigma^2 L}}\right) - xF\left(\frac{R + Q - x - DL}{\sqrt{\sigma^2 L}}\right)\right)dx$

$$+\frac{1}{Q}\int_R^{R+Q}\left(x - xF\left(\frac{R + Q - x - DL}{\sqrt{\sigma^2 L}}\right)\right)dx$$

Simplifying

$$D(Q,\ R) = \frac{1}{Q}\int_0^R xF\left(\frac{R - x - DL}{\sqrt{\sigma^2 L}}\right)dx - \frac{1}{Q}\int_0^{R+Q} xF\left(\frac{R + Q - x - DL}{\sqrt{\sigma^2 L}}\right)dx$$

$$+\frac{1}{Q}\int_R^{R+Q} xdx$$

Let V= $\frac{R - x - DL}{\sqrt{\sigma^2 L}}$ and $\frac{R + Q - x - DL}{\sqrt{\sigma^2 L}}$ In each integral

Then $D(Q_1 R) = \frac{\sqrt{\sigma^2 L}}{Q}\int_{R\frac{\sqrt{R-DL}}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} \sqrt{\sigma^2 L}V - R + DL)F(V)dV$

$$+\frac{\sqrt{\sigma^2 L}}{Q}\int_{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} (\sigma^2 LV - R - Q + DL)dV + \frac{(R+Q)^2 - R^2}{2Q}$$

Remembering that $k = \dfrac{R - DL}{\sqrt{\sigma^2 L}}$

Then expressing R in terms of standard deviations of stock

$$D(Q,K) = -\frac{\sigma^2 L}{Q}\int_k^{\frac{-DL}{\sqrt{\sigma^2 L}}} (V - k)F(V)dV + \frac{(\sigma^2 L)}{Q}\int_{\frac{k+Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}}$$

$$(V - (k + \frac{Q}{\sqrt{\sigma^2 L}})F(V)dV + \frac{Q}{2} + K\sqrt{\sigma^2 L}$$

Simplifying

$$D(Q,K) = -\frac{\sigma^2 L}{Q}\int_k^{\frac{-DL}{\sqrt{\sigma^2 L}}} VF(V)dV + \frac{(\sigma^2 L)}{Q}\int_k^{\frac{-DL}{\sqrt{\sigma^2 L}}} kF(V)dV$$

$$+\frac{\sigma^2 L}{Q}\int_{\frac{k+Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} vF(V)dv + \frac{(\sigma^2 L)}{Q}\int_{\frac{k+Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} (k + \frac{Q}{\sqrt{\sigma^2 L}})F(V)dV$$

$$+\frac{Q}{2} + k\sqrt{\sigma^2 L}$$

Hence D(Q, k), noting that F $\left(\dfrac{-DL}{\sqrt{\sigma^2 L}}\right) = 1$

and $\int_k^{\infty} F(v)dv = -kF(k) + g(k)$

and $\int_k^{\alpha} vF(v)dv = 1/2(1 - k^2)F(k) + kg(k)$

Hence D(Q, k)

$$= -\frac{\sigma^2 L}{Q}\left(k(F(k) - g(k) - \left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) + g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\right) - \frac{\sigma^2 L}{2Q}\left((1 - k^2)F(k) + kg(k)\right)$$

$$-\left(1 - \left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)^2 F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{\sigma^2 L}{Q}\left(\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)F\left(K + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\left(k + \frac{Q}{\sqrt{2\sigma^2 L}}\right)g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) - \left(1 - \frac{DL}{\sqrt{\sigma^2 L}}\right)^2 F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\Bigg) + \frac{Q}{2} + k\sqrt{\sigma^2 L}$$

Which gives

$$D(Q,k) = k\sqrt{\sigma^2 L} + \frac{Q}{2} + \frac{\sigma^2 L}{2}\left((1 + k^2)\overline{(k)} - kg(k)\right)$$

$$-\frac{\sigma^2 L}{2}\left(1 + \frac{(k+Q)^2}{\sigma^2 L}\right)F\left(\frac{k+Q}{\sqrt{\sigma^2 L}}\right)$$

$$-\left(k+\frac{Q}{\sqrt{\sigma^2L}}\right)g\left(\frac{k+Q}{\sqrt{\sigma^2L}}\right)\Bigg) \quad \ldots\ldots(25)$$

Remembering that k = $\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$ then

Substituting for k then

$$D(Q,R,L)=\frac{Q}{2}+R-DL+\frac{\sigma^2L}{2}\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)^2\right)$$

$$F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\frac{\sigma^2L}{2}\left(1+\frac{R+Q-DL}{\sqrt{\sigma^2L}}\right)^2$$

$$F\left(\frac{R+Q-DL}{\sqrt{\sigma^2L}}\right)+\left(\frac{R+Q-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{R+Q-DL}{\sqrt{\sigma^2L}}\right)$$

But $\frac{\sigma^2L}{2}\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)^2\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$ ……………….…(26)

$$=\int_R^\infty\left(\sqrt{\sigma^2L}g\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)\right)-(V-DL)F\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)\Bigg)dV \quad \ldots\ldots(27)$$

Hence

$$D(Q,R,L)=\frac{Q}{2}+R-DL+\int_R^{R+Q}\sqrt{\sigma^2L}g\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)$$

$$-(V-DL)F\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)dV$$

Where D(Q,R,L) is the inventory on hand for a given lead time L.
Hence expected on hand inventory

$$=\int_0^\infty H(L)D(Q,R,L)DL \quad \ldots\ldots(28)$$

$$\int_0^\infty\left(\frac{Q}{2}+R\right)H(L)DL-D\int_0^\infty LH(L)dL+\frac{1}{Q}\int_0^\infty\int_R^{R+Q}H(L)\sqrt{\sigma^2L}$$

$$\left(g\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)-(V-DL)F\left(\frac{V-DL}{\sqrt{\sigma^2L}}\right)dVdL\right) \quad \ldots\ldots(29)$$

Noting that

$\int_0^{\alpha}\int_R^{R+Q} H(L)\sqrt{\sigma^2 L}\left(g\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right) - (V-DL)F\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right)\right) dVdL$ is the coefficient of b_2 in (iv)

Then integrating (29)
We have

$$D(Q,R) = \frac{Q}{2} + R - \frac{Dk}{\alpha} + \frac{1}{Q}\left(G_4(R) - G_4(R+Q)\right)$$

Cost of ordering = $\left(\frac{DS}{Q}\right)$

Inventory costs for mode (Q_1R)

$$C = \frac{DS}{Q} + hcD(Q_1R) + \frac{b_1}{Q}\left(G_4\left(R + Dp + \frac{1}{Q}(G_4(R) - G_4(R+Q)\right)\right)$$

Cost of ordering = $\frac{DS}{Q}$ ………………………………………………………….(30)

Inventory costs for mode (Q_1R)

$$C = \frac{DS}{Q} + hcD(Q_1R) + \frac{b_1}{Q}\left(G_3(R+Dp) - G_3(R+Q+Dp)\right)$$

$$+\frac{b_2}{Q}\left(G_4(R+Dp) - G_4(R+Q+Dp)\right) + b_3(G_5(R+Dp) - G_5(R+Q+Dp)) \text{…….(31)}$$

Substituting for D(Q_1R) from (30)

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc(R - \frac{Dk}{\alpha}) + b_1\left(G_3(R+Dp) - G_3(R+Q+Dp)\right)$$

$$+\frac{b_2}{Q}\left(G_4(R+Dp) - G_4(R+Q+Dp)\right) + b_3(G_5(R+Dp) - G_5(R+Q+Dp))$$

$$+\frac{hc}{Q}(G_4(R) - G_4(R+Q) \text{…………………………………….…….(34)}$$

The corresponding cost when no period of grace p is operating is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc(R - \frac{Dk}{\alpha}) + b_1\left(G_3(R) - G_3(R+Q)\right)$$

$$+\frac{(hc + b_2)}{Q}(G_4(R) - G_4(R+Q)) + \frac{b_3}{Q}(G_5(R) - G_5(R+Q))$$

IV. IMPACT OF THE STUDY

The study will enable industries or organizations having thousands of items in their warehouses located in various locations of the world and items supplied by different manufacturers with accurately use realistic lead times in arriving at their inventory cost.
In many of such cases lead times are not constant.
Expressing lead time as continuous gives a more realistic estimate of inventory costs.

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