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On analytical solutions for the nonlinear diffusion equation

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ABSTRACT. *The nonlinear diffusion equation arises in many important areas of nonlinear problems of heat and mass transfer, biological systems and processes involving fluid flow and most of the known exact solutions turn out to be approximate solutions in the form of a series which is the exact solution in the closed form. The approximate results obtained by using Homotopy perturbation transform method (HPTM) and have been compared with the exact solutions by using software "mathematica" to show the stability of the solutions of nonlinear equation. The comparisons indicate that there is a very good agreement between the HPTM solutions and exact solutions in terms of accuracy.*

KEYWORDS: *Laplace transform method, He's polynomial, Homotopy perturbation method, Nonlinear diffusion equation*

1. INTRODUCTION

The nonlinear diffusion equation (1) is a prominent example of porous medium equation [1]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) \text{ with } D(u) = u^m \quad (1)$$

where m is a rational number, x and t denote derivatives with respect to space and time, when they are used as subscripts. This equation is one of the simplest examples of nonlinear evolution equation of parabolic type. It appears in the description of different natural phenomena [2-5] such as heat transfer or diffusion, processes involving fluid flow, may be the best known of them is the best known of them is the description of the flow of an isentropic gas through a porous medium, modelled independently by Leibenzon and Muskat around 1930. An earlier application is found in the study of groundwater infiltration by Boussisneq in 1903 and the particular cases $m = 2$, it leads to Boussinesq equation is used in the field of buoyancy-driven flow (also known as natural convection), $m = 3$, the equation (1) can be obtained from the Navier-Stokes equations. Equation (1) has also applications to many physical systems including the fluid dynamics of thin films [6].

Most phenomena described by nonlinear equations are still difficult to obtain accurate results and often more difficult to get an analytic approximation than a numerical one. The results are obtained by some techniques as Adomian's decomposition method, the variational iteration method, the weighted finite difference method, the Laplace decomposition method and the variational iteration decomposition method are divergent in most cases and which results in causing a lot of chaos. These methods have their own limitation like the calculation of Adomain's polynomials and the Lagrange's multipliers.

In this work, to overcome these difficulties and drawbacks such new technique, which is called Homotopy perturbation transform method (HPTM) and using the "software mathematica" are introduced for finding the approximate results with different powers of m .

2. Homotopy Perturbation Transform Method (HPTM)

To illustrate the basic idea of this method [7, 8], we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the formula:

$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t)$ (2) $u(x,0) = h(x)$, $u_t(x,0) = f(x)$ (3)

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D , N represent the general non-linear differential operator and $g(x,t)$ is the source term. Taking the Laplace transform (denoted by L) on both side of Eqs. (2) and (3):

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (4)$$

Using the differentiation property of the Laplace transform, [9] we have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2}L[Ru(x,t)] - \frac{1}{s^2}L[Nu(x,t)] + \frac{1}{s^2}L[g(x,t)] \quad (5)$$

Operating with the Laplaceinverse on both side of Eq. (5) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right] \quad (6)$$

Where $G(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method [10, 11]

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \qquad (7)$$

And the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \qquad (8)$$

For some He's polynomial H_n that are given by

$$H_n(u_0 \ldots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}(p^i u_i)\right)\right]_{p=0}, \quad n = 0,1,2,3,\ldots$$

Substituting Eqs. (7) and (8) in Eq. (6) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \quad (9)$$

This is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomial.

Comparing the coefficient of like powers of p, the following approximations are obtained [12]

$$p^0 : u_0(x,t) = -\frac{1}{s^2} L\left[R u_0(x,t) + H_0(u) \right]$$

$$p^1 : u_1(x,t) = -\frac{1}{s^2} L\left[R u_1(x,t) + H_0(u) \right] \quad (10)$$

$$p^2 : u_2(x,t) = -\frac{1}{s^2} L\left[R u_1(x,t) + H_1(u) \right]$$

$$p^3 : u_3(x,t) = -\frac{1}{s^2} L\left[R u_2(x,t) + H_2(u) \right]$$

The best approximations for the solutions are

$$u = \lim_{p \to 1} u_n = u_0 + u_1 + u_2 + \ldots (11)$$

3. Resolution and results

In order to assess the accuracy of solution HPTM for equation (1), we will consider the three following examples.

3.1. Example 1. Let us take $m = 1$, Eq. (1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x} \right) (12)$$

with initial condition $u(x,0) = x$

3.1.1. Homotopy perturbation transform method

Apply Laplace transform on both the sides of Eq. (12) subject to the initial condition

$$L\left[\frac{\partial u}{\partial t} \right] = L\left[\left(\frac{\partial u}{\partial x} \right)^2 \right] + L\left[u \frac{\partial^2 u}{\partial x^2} \right] (13)$$

This can be written on applying the above specified initial condition as

$$u(x,t) = \frac{1}{s}(x) + \frac{1}{s} L\left[\left(\frac{\partial u}{\partial x} \right)^2 \right] + \frac{1}{s} L\left[\left(u \frac{\partial^2 u}{\partial x^2} \right) \right] (14)$$

Taking inverse Laplace Transform on both sides, we get

$$L^{-1}\left[u(x,s) \right] = xL^{-1}\left[\frac{1}{s} \right] + L^{-1}\left[\frac{1}{s} L\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(u \frac{\partial^2 u}{\partial x^2} \right) \right] \right] \quad (15)$$

$u(x,t) = x + L^{-1}\left[\frac{1}{s} L\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(u \frac{\partial^2 u}{\partial x^2} \right) \right] \right]$ Now on applying the homotopy perturbation method in the form

$$u(x,t) = \sum_{n=0}^{\infty} p^n (u_n(x,t)) \quad (16)$$

Equation (15) can be reduces to

$$\sum_{n=0}^{\infty} p^n (u_n(x,t)) = x + L^{-1}\left[\frac{1}{s} L\left[\begin{array}{l} \left(\left(\sum_{n=0}^{\infty} p^n (u(x,t)) \right)_x \right)^2 + \\ \left(\left(\sum_{n=0}^{\infty} p^n (u_n(x,t)) \right) \left(\sum_{n=0}^{\infty} p^n (u_n(x,t)) \right)_{xx} \right) \end{array} \right] \right] \quad (17)$$

On expansion of equation (17) and comparing the coefficient of various powers of p, we get

$$p^0 : u_0(x,t) = x$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right] + L^{-1}\left[\frac{1}{s}L\left[\left(u_0\frac{\partial^2 u_0}{\partial x^2}\right)\right]\right] \quad (18)$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s}L\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right]\right] + L^{-1}\left[\frac{1}{s}L\left[\left(u_1\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial^2 u_1}{\partial x^2}\right)\right]\right]$$

In this case the values obtained as $u_0 = x$, $u_1 = t$ and $u_2 = 0$ which follows $u_n(x,t) = 0$ for $n \geq 2$. Putting these values in (11) we get the solution as

$$u(x,t) = x + t \quad (19)$$

This is same as the exact solution given in [13].

3.2. Example 2. Let us take $m = -1$ in equation (1), we get

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((u)^{-1}\frac{\partial u}{\partial x}\right) \quad (20)$$

With initial condition as $u(x,0) = \frac{1}{x}$

Exact solution of this equation is

$$u(x,t) = \frac{1}{x-t} \quad (21)$$

3.2.1. Homotopy perturbation transform method

Using HPTM we can find solution by applying Laplace transform on both side of equation (20) subject to the initial condition

$$L\left[\frac{\partial u}{\partial t}\right] = L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right] - L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right] \quad (22)$$

This can be written as

$$\left[su(x,s) - u(x,0)\right] = L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right] - L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right] \quad (23)$$

On applying the above specified initial condition we get

$$su(x,s) - \left(\frac{1}{x}\right) = L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right] - L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right] \quad (24)$$

$$u(x,s) = \frac{1}{s}\left(\frac{1}{x}\right) + \frac{1}{s}L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right] - \frac{1}{s}L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right] \quad (25)$$

Taking Inverse Laplace Transform on both sides we get

$$L^{-1}\left[u(x,s)\right] = L^{-1}\left[\frac{1}{s}L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right]\right] - L^{-1}\left[\frac{1}{s}L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right]\right] \quad (26)$$

$$u(x,t) = L^{-1}\left[\frac{1}{s}L\left[(u^{-1})\left(\frac{\partial^2 u}{\partial x^2}\right)\right]\right] - L^{-1}\left[\frac{1}{s}L\left[(u^{-2})\left(\frac{\partial u}{\partial x}\right)^2\right]\right] \quad (27)$$

Now we apply the homotopy perturbation method in the form

$$u(x,t)=\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right) \quad (28)$$

Using Binomial expansion and He's Approximation, equation (26) reduces to

$$\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right) = L^{-1}\left[\frac{1}{s}L\left[\left(\left(\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right)\right)^{-1}\right)\left(\left(\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right)\right)_{xx}\right)\right]\right]$$

$$-L^{-1}\left[\frac{1}{s}L\left[\left(\left(\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right)\right)^{-2}\right)\left(\left(\sum_{n=0}^{\infty} p^n \left(u_n(x,t)\right)\right)_{x}\right)^{2}\right]\right] \quad (29)$$

This can be written in expanded form as

$$u_0+pu_1+p^2u_2+\ldots=\left(\frac{1}{x}\right)+pL^{-1}\left[\frac{1}{s}L\left[\begin{array}{l}\left(\left(u_0+pu_1+p^2u_2+\ldots\right)\right)^{-1}\\ \left(\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p^2\frac{\partial^2 u_2}{\partial x^2}+\ldots\right)\end{array}\right]\right]$$

$$-pL^{-1}\left[\frac{1}{s}L\left[\left(\left(u_0+pu_1+p^2u_2+\ldots\right)\right)^{-2}\left(\left(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+\ldots\right)^2\right)\right]\right] \quad (30)$$

Comparing the coefficient of various power of p, we get

$$p^0: u_0(x,t)=\frac{1}{x}$$

$$p^1: u_1(x,t)=L^{-1}\left[\frac{1}{s}L\left[\left(u_0\right)^{-1}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right]\right]-L^{-1}\left[\frac{1}{s}L\left[\left(\left(u_0\right)^{-2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right]\right] \quad (31)$$

$$p^2: u_2(x,t)=L^{-1}\left[\frac{1}{s}L\left[\left(u_0\right)^{-1}\left(\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_0}{\partial x^2}\left(\frac{u_1}{u_0}\right)\right)\right]\right]-L^{-1}\left[\frac{1}{s}L\left(u_0\right)^{-2}\left(2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)-2\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{u_1}{u_0}\right)\right)\right]$$

Proceeding in similar manner we can obtain further values, substituting above values in equation (11) we get solution in the form of a series

$$u(x,t)=\frac{1}{x}+\frac{t}{x^2}+\frac{t^2}{x^3}+\frac{t^3}{x^4}+\ldots \quad (32)$$

Which is the exact solution obtained in (21) in the closed form.

3.3. Example. Let us take $m=-4/3$, equation (1) becomes

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right) \quad (33)$$

The exact solution to Eq. (33) is given by [14]

$$u(x,t)=(2x-3t)^{-3/4} \quad (34)$$

3.3.1. Homotopyperturbation transform method

Applying the same procedure again on (33) subject to the initial condition

$u(x,0)=(2x)^{-3/4}$ (35)

We get,

$$p^{0}: u_{0}(x,t)=(2x)^{-3/4}$$

$$p^{1}: u_{1}(x,t)=L^{-1}\left[\frac{1}{s}L\left[(u_{0})^{-4/3}\left(\frac{\partial^{2}u_{0}}{\partial x^{2}}\right)\right]\right]-L^{-1}\left[\frac{1}{s}L\left[\left(\frac{4}{3}\right)(u_{0})^{-7/3}\left(\frac{\partial u_{0}}{\partial x}\right)^{2}\right]\right] \quad (36)$$

$$p^{2}: u_{2}(x,t)=L^{-1}\left[\frac{1}{s}L\left[(u_{0})^{-4/3}\left(\frac{\partial^{2}u_{1}}{\partial x^{2}}-\left(\frac{4}{3}\right)\frac{\partial^{2}u_{0}}{\partial x^{2}}\left(\frac{u_{1}}{u_{0}}\right)\right)\right]\right]$$

$$-L^{-1}\left[\frac{1}{s}L\left[\left(\frac{4}{3}\right)(u_{0})^{-7/3}\left(2\left(\frac{\partial u_{0}}{\partial x}\right)\left(\frac{\partial u_{1}}{\partial x}\right)-\frac{7}{3}\left(\frac{\partial u_{0}}{\partial x}\right)^{2}\left(\frac{u_{1}}{u_{0}}\right)\right)\right]\right]$$

On solving above we get values as $u_{0}=(2x)^{-3/4}, u_{1}=9\times 2^{-15/4}\times x^{-7/4}\times t, u_{2}=189\times 2^{-31/4}\times x^{-11/4}\times t^{2}$ and so on. Substituting these terms in Eq. (11), one obtains

$$u(x,t)=u_{0}(x,t)+u_{1}(x,t)+u_{2}(x,t)+\ldots$$

$$u(x,t)=(2x)^{-3/4}+9\times 2^{-15/4}x^{-7/4}t+189\times 2^{-31/4}x^{-11/4}t^{2}+\ldots \quad (37)$$

This gives the exact solution obtained in Eq. (34) in the closed form.

4. Comparing the HPTM results with the exact solution

The approximate and plotted results of the example 1 are required to obtain accurate solution. Both the exact solutions and the approximate solutions are plotted in Figs.4.2-4.5 and only few terms are required to obtain accurate solution.

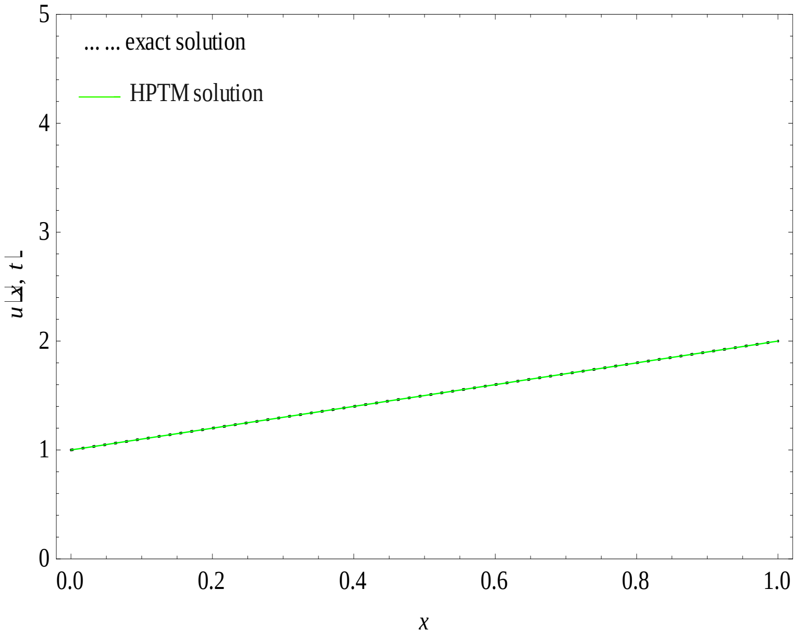


Fig.4.1.The comparison of the exact solution and HPTM solution for example1 at $t = 1$

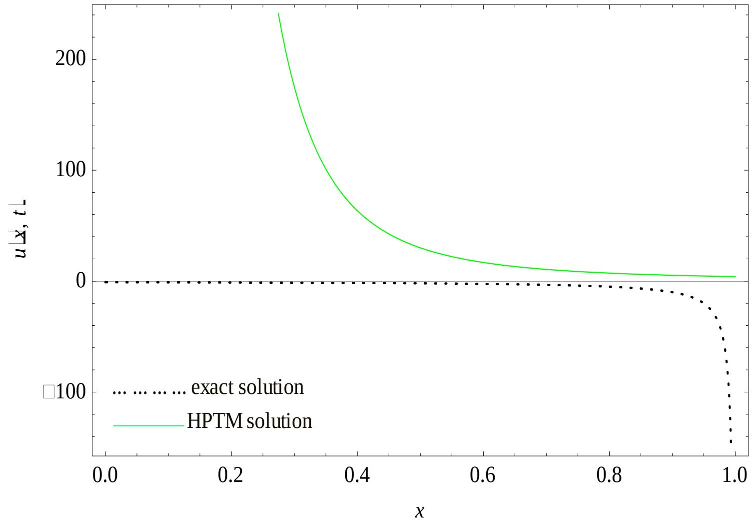


Fig.4.2.The comparison of the exact solution and the 3[th] order HPTMsolutionfor example 2 at $t = 1$

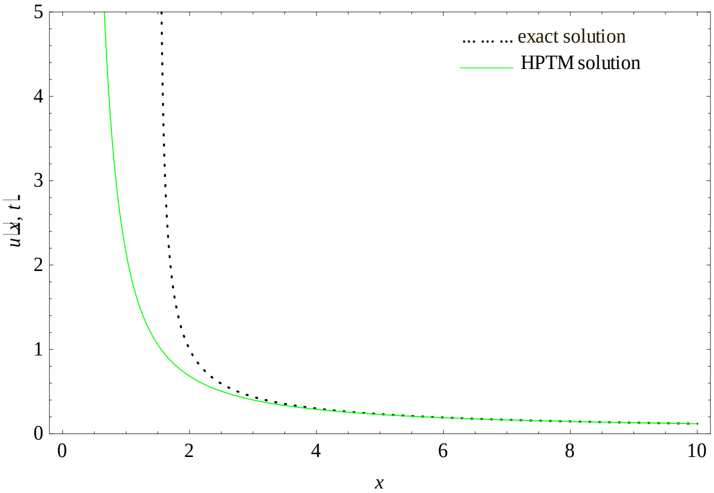
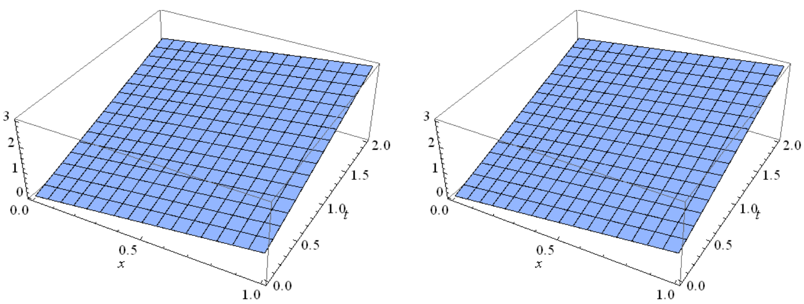
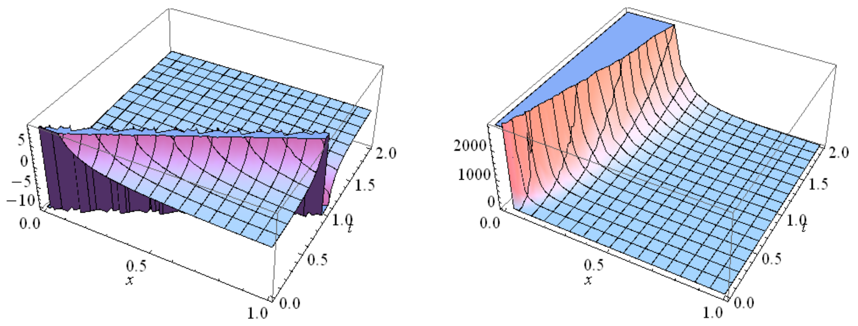


Fig.4.3.The comparison of the exact solution and the 2[th]orderHPTM solution for example 3 at $t = 1$



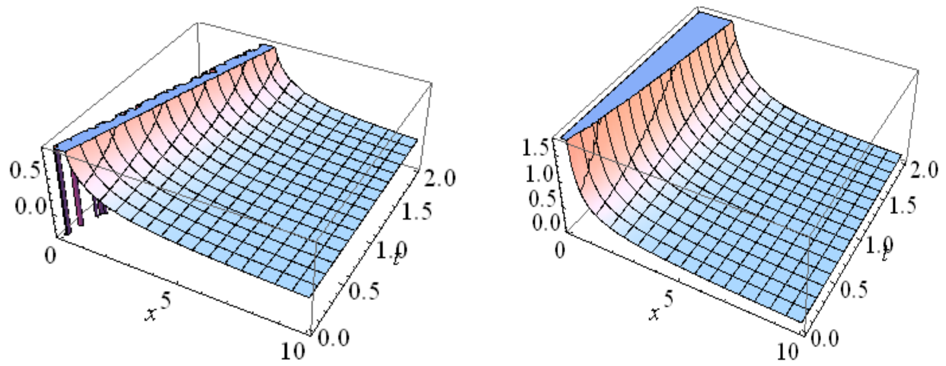
a. Exact solution b.HPTM solution

Fig.4.4.Comparison between the exact solution and HPTM solution of $u(x,t)$ for $m = 1$



a.Exact solution b.HPTM solution

Fig.4.5. Comparison between the exact solution and the HPTM solution of $u(x,t)$ for $m=-1$



a.Exact solution b.HPTM solution

Fig.4.6. Comparison between the exact solution and the HPTM solution of $u(x,t)$ for $m=-4/3$

4. Conclusion

In this paper, the solutions for nonlinear diffusion equation obtained with different powers of m by the homotopyperturbation transform method are an infinite power series for appropriate initial condition, which are the exact solution in the closed form. The results (see Figs.4.1-4.5) show accuracy between HPTM solution and exact solution for different powers of m because in many cases an exact solution in a rapidly convergent sequence with elegantly computed terms. Finally, we conclude that the nonlinear problems have the accurate solution.

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