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Approximate Solution of the Dirichlet Problems in Polar Co-Ordinates

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Abstract: - This paper is concerned with the investigation of the Helmholtz type equation with the Dirichlet boundary conditions in polar co-ordinates .We present a numerical method for solving this equation and obtain the matrix form of equations. For our purpose define the mesh points in the $r-\theta$ plane by the points of intersection of the circles $r = ih, (i = 1, \dots, n)$ and the straight lines $\theta = j\delta\theta, j = 0,1,2, \dots$

Keywords: *- finite difference, circular boundary, polar co-ordinates, Cartesian co-ordinates, curved boundary.*

I. INTRODUCTION

Many physical problems involve solving elliptic equations with circular boundaries. Finite difference problems involving circular boundaries usually are solved more conveniently in polar co-ordinates than Cartesian co-ordinates. In this case, we first transform the rectangular coordinate system into the convenient polar or cylindrical coordinates. In the present paper, we consider Helmholtz equation

$$U_{xx} + U_{yy} + \lambda U = F(x, y)$$

With the Dirichlet boundary conditions on Ω, where λ is a positive real constant. The Helmholtz equation or reduced wave equation is an elliptic partial differential equation. It takes its name from the German physicist Hermann Helmholtz (1821-1894), a researcher in acoustics, electromagnetism, and physiology. This equation occurs when we are looking for mono frequency or time harmonic solutions for the wave equation.

A. S. Fokas introduced a new method for solving boundary value problems for linear and for integrable nonlinear PDEs [1]. Daniel ben-Avraham and Athanassios S. Fokas applied this method to the Helmholtz equation [2]. We want to solve this problem in polar co-ordinates. K. Mohseni and T. Colonius presented a numerical treatment of polar coordinate singularities [4]. There are other methods to solve pure problems in polar or cylindrical coordinates [6, 3]. In the next section, we present a finite difference scheme for solution of Helmholtz equation.

II. FINITE DIFFERENCE SCHEME

Let us consider E.q (1) on $\Omega = \{(x, y) | x^2 + y^2 < 1\}$ with the Dirichlet $U = g$ boundary conditions on Ω. Note that if $\lambda = 0$, the Helmholtz differential equation reduces to Laplace equation $U_{xx} + U_{yy} = F(x, y)$ In this paper, we only consider those solutions U of (1) which are defined and analytic in the real variables x, y for domain Ω in the plane R^2. By using the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$ where $r = (x^2 + y^2)^{1/2}$ and $\theta = \arctan\frac{y}{x}$, and setting $u(r, \theta) = U(x, y)$ and $f(r, \theta) = F(x, y)$ E.q (1) becomes:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \lambda u = f(r, \theta) \qquad 0 < r < 1\,, 0 < \theta < 2\pi$$

For non-zero values of r there is no problem, but at $r = 0$ the right side appears to contain singularities. In this case, we can replace the polar co-ordinate form of equation by its Cartesian equivalent. In the present paper, we choose a grid which the grid points are in the $r - \theta$ plane as follow:

$$r_i = \frac{2i + 1}{2}\Delta r \qquad i = 0, 1, \dots, n + 1$$

$$\theta_j = j\Delta\theta \qquad j = 0,1,\dots,m+1$$

The notations u_{ij}, f_{ij} and g_j are used for the finite difference approximations of $u(r_i,\theta_j)$, $f(r_i,\theta_j)$and $g(\theta_j)$ respectively.

By using the central difference scheme for E.q (1) we have:

$$\left(1-\frac{1}{2i+1}\right)u_{i-1j} + a_i u_{ij}\left(1+\frac{1}{2i+1}\right)u_{i+1j} + \frac{4}{(2i+1)^2(\Delta\theta)^2}u_{ij-1} + \frac{4}{(2i+1)^2(\Delta\theta)^2}u_{ij+1} = f(r_i,\theta_j)$$

where $a_i = \lambda - 2 - \frac{8}{(2i+1)^2(\Delta\theta)^2}$. By

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{im} \end{bmatrix} \qquad i = 1,\dots,n$$

The linear system of equations is as follows:

$$AU = B$$

where A is a $mn \times mn$ matrix which can be written in partitioned form as:

$$A = \begin{pmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & A_n & B_n \end{pmatrix}$$

for $i = 1,2,\dots,n$ we have:

$$B_i = \begin{pmatrix} a_i & \frac{4}{(2i+1)^2(\Delta\theta)^2} & & \\ \frac{4}{(2i+1)^2(\Delta\theta)^2} & a_i & \frac{4}{(2i+1)^2(\Delta\theta)^2} & \\ \ddots & \ddots & \ddots & \\ & & \frac{4}{(2i+1)^2(\Delta\theta)^2} & a_i \end{pmatrix}$$

And $A_i = (1-\frac{1}{2i+1})I_{m\times m}$ and $C_i = (1+\frac{1}{2i+1})I_{m\times m}$. B also is a column vector determined by the boundary values as follow:

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where

$$b_1 = \begin{pmatrix} f(r_1,\theta_1) - \left(1-\frac{1}{3}\right)u_{01} - \frac{4}{9(\Delta\theta)^2}u_{10} \\ f(r_1,\theta_2) - \left(1-\frac{1}{3}\right)u_{02} \\ \vdots \\ f(r_1,\theta_m) - \left(1-\frac{1}{3}\right)u_{0m} - \frac{4}{9(\Delta\theta)^2}u_{1m+1} \end{pmatrix}, \quad b_i = \begin{pmatrix} f(r_i,\theta_1) - \frac{4}{(2i+1)^2 9(\Delta\theta)^2}u_{i0} \\ f(r_i,\theta_2) \\ \vdots \\ f(r_i,\theta_m) - \frac{4}{(2i+1)^2(\Delta\theta)^2}u_{im+1} \end{pmatrix}$$

where $i = 2,\dots,n-1$ and

$$b_1 = \begin{pmatrix} f(r_n,\theta_1) - \left(1+\frac{1}{2n+1}\right)u_{n+1,1} - \frac{4}{(2n+1)^2(\Delta\theta)^2}u_{n,0} \\ f(r_n,\theta_2) - \left(1+\frac{1}{2n+1}\right)u_{n+1,2} \\ \vdots \\ f(r_n,\theta_m) - \left(1+\frac{1}{2n+1}\right)u_{n+1,m} - \frac{4}{(2n+1)^2(\Delta\theta)^2}u_{n,m+1} \end{pmatrix}$$

A is a tridiagonal and invertible matrix, therefore this system has a unique solution. There are a lot of methods for solution these linear system equations; you can see [5].

III. CONCLUSION

Our purpose in this article is solving the Dirichlet problems for the Helmholtz equation. Here, a numerical method for solving this problem is investigated. We first obtain Helmholtz equation in polar coordinate. Afterwards, we present an implicit scheme and obtain the matrix form of this equation. Finally, the obtained system can be solved by various methods.

REFERENCES

[1] A. S. Fokas, On the integrability of linear and nonlinear PDEs, J. Math. Phys. 41 (2000) 4188.
[2] Daniel ben-Avraham and Athanassios S. Fokas, The solution of the modified Helmholtz equation in a triangular domain and an application to diffusion-limited coalescence, Phys. Lett. A. 263 (1999) 355-359.
[3] K. Fukagata, N. Kasagi, Highly energy-conservative finite difference method for the cylindrical coordinate system, J. Comput. Phys. 181 (2002) 478-498.
[4] K. Mohseni, T.Colonius, Numerical treatment of polar coordinate singularities, J. Comput. Phys. 151 (2000) 787-795.
[5] R. A. Sweet, A cyclic reduction algorithm for block tridiagonal systems of arbitrary dimension, SIAM J. Numer. Anal. 14 (1977) 706.
[6] R. Verzicco, P.Orlandi, A finite-difference scheme for three-dimensional incompressible flows in cylindrical coordinates, J. Comput. Phys. 123 (1996) 402-414.