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Families of Norms Generated By 2-Norm

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Abstract: *- In* [1] *S. Gähler proof that for any linearly independent vector* $a,b\in L$, *the equality* $\|x\|=\|x,a\|+\|x,b\|$, $x\in L$ *defines a norm on* L. *This result is generalized by A. Misiak in* [2], *and in* [3] *is presented other proof of this result. Moreover, H. Gunawan in* [4] *generalized these results. In this paper we'll generalize the S. Gähler's result of 2-normed space, which can easy be generalized on n-normed space.*
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I. INTRODUCTION

Let L be a real vector space with dimension greater than 1 and $\|\cdot,\cdot\|$ be a real function on $L\times L$ such that:

a) $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent ;

b) $\|x,y\|=\|y,x\|$, for every $x,y\in L$;

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for every $x,y\in L$ and for every $\alpha\in\mathbf{R}$;

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for every $x,y,z\in L.$

The function $\|\cdot,\cdot\|$ is called 2-*norm* on L, a $(L,\|\cdot,\cdot\|)$ is called *vector* 2-*normed space* ([1]). Some of the basic properties of a 2-norm are that it's nonnegative, i.e.

$$\|x,y\|\geq 0,\text{ for every } x,y\in L$$

and

$$\|x,y+\alpha x\|=\|x,y\|,\text{ for every } x,y\in L \text{ and for every } \alpha\in\mathbf{R}.$$

Let $n>1$ be a real number, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function on $L\times L\times L$ which satisfies the following conditions:

i) $(x,x|y)\geq 0$, for every $x,y\in L$ и $(x,x|y)=0$ if and only if x and y are linearly dependent;

ii) $(x,y|z)=(y,x|z)$, for every $x,y,z\in L$;

iii) $(x,x|y)=(y,y|x)$, for every $x,y\in L$;

iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for every $x,y,z\in L$ for every and $\alpha\in\mathbf{R}$; and

v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for every $x,x_1,y,z\in L$.

The function $(\cdot,\cdot|\cdot)$ is called 2-*inner product*, and $(L,(\cdot,\cdot|\cdot))$ is called 2-*pre-Hilbert space* ([5]).

Concepts of 2-norm and 2-inner product are two-dimensional analogies of concepts of norm and inner product. R. Ehret proved ([5]) that, if $(L,(\cdot,\cdot|\cdot))$ be 2-pre-Hilbert space, than

$$\|x,y\|=(x,x|y)^{1/2},\quad x,y\in L \tag{1}$$

defines 2-norm. So, we get vector 2-normed space $(L,\|\cdot,\cdot\|)$ and for each $x,y,z\in L$ the following equalities are true:

$$(x,y|z)=\frac{\|x+y,z\|^2-\|x-y,z\|^2}{4}, \tag{2}$$

$$\|x+y,z\|^2+\|x-y,z\|^2=2(\|x,z\|^2+\|y,z\|^2), \tag{3}$$

In fact, the equality (3) is two-dimensional analogy of parallelogram equality and is called parallelepiped equality. Further, if $(L,\|\cdot,\cdot\|)$ is vector 2-normed space such that (1) is satisfied for every $x,y,z\in L$, then (3) defines 2-inner product on L, and moreover the equality (2) is satisfied.

II. NORMS DEFINED BY 2-NORM

Theorem 1. Let $(L,\|\cdot,\cdot\|)$, $p\geq 1$ and $\{a,b\}$ be linear independent subset of L. Then,

$$\|x\|=(\|x,a\|^p+\|x,b\|^p)^{1/p},\ x\in L \quad (4)$$

define norm of L.

Proof. It's clear that, $\|x\|\geq 0$ and $\|0\|=0$. Letting $\|x\|=0$ in (5) we get that $\|x,a\|=\|x,b\|=0$. According the definition of 2-norm, we can conclude that the sets $\{x,a\}$ and $\{x,b\}$ are linearly dependent. The fact that the set $\{a,b\}$ is linearly independent implies $tx=\alpha a$ and $qx=\beta b$, for some $t,q\neq 0$. So, $\alpha qa=\beta tb$ and $\{a,b\}$ is linearly independent set and also $t,q\neq 0$. The last equality and the conditions mentioned above, implies $\alpha=\beta=0$, i.e. $x=0$. Let $x\in L$ and $\alpha\in\mathbf{R}$, then (5) implies the following

$$\|\alpha x\|=(\|\alpha x,a\|^p+\|\alpha x,b\|^p)^{1/p}=|\alpha|(\|x,a\|^p+\|x,b\|^p)^{1/p}=|\alpha|\cdot\|x\|.$$

Finally, using parallelepiped inequality and Minkovski's inequality we get that for each $x,y\in L$ it's true that

$$\begin{aligned}\|x+y\|&=(\|x+y,a\|^p+\|x+y,b\|^p)^{1/p}\\&\leq[(\|x,a\|+\|y,a\|)^p+(\|x,b\|+\|y,b\|)^p]^{1/p}\\&\leq(\|x,a\|^p+\|x,a\|^p)^{1/p}+(\|y,a\|^p+\|y,b\|^p)^{1/p}\\&=\|x\|+\|y\|.\end{aligned}$$

It means that (4) define norm of L, which will be denoted as $\|\cdot\|_{a,b,p}$. ■

Theorem 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space and $\{a,b\}$ be linearly independent subset of L. Then

$$\|x\|=\max\{\|x,a\|,\|x,b\|\},\ x\in L \quad (5)$$

defines norm of L.

Proof. Clearly, $\|x\|\geq 0$ and $\|0\|=0$. Let $\|x\|=0$. Then (5) implies $\|x,a\|=\|x,b\|=0$, and analogously as in the proof of the theorem 1 we get that $x=0$. Let $x\in L$ and $\alpha\in\mathbf{R}$. The equality (5) implies

$$\|\alpha x\|=\max(\|\alpha x,a\|,\|\alpha x,b\|\}=\max\{|\alpha|\cdot\|x,a\|,|\alpha|\cdot\|x,b\|\}=|\alpha|\cdot\|x\|.$$

Further, using the properties of maximum and the parallelepiped inequality we get the following

$$\begin{aligned}\|x+y\|&=\max\{\|x+y,a\|,\|x+y,b\|\}\\&\leq\max\{\|x,a\|+\|y,a\|,\|x,b\|+\|y,b\|\}\\&\leq\max\{\|x,a\|,\|x,b\|\}+\max\{\|y,a\|,\|y,b\|\}\\&=\|x\|+\|y\|.\end{aligned}$$

It means that (5) defines norm of L, which will be denoted as $\|\cdot\|_{a,b,\infty}$. ■

Theorem 3. Let $(L,\|\cdot,\cdot\|)$ be 2-normed space and $\{a,b\}$ is linearly independent subset of L. Then, for every $p,q\geq 1$ the $\|\cdot\|_{a,b,p}$, $\|\cdot\|_{a,b,q}$ and $\|\cdot\|_{a,b,\infty}$ are equivalent.

Proof. Let $p\geq 1$. Then, for every $x\in L$,

$$\begin{aligned}\|x\|_{a,b,\infty}&=\max\{\|x,a\|,\|x,b\|\}\leq(\|x,a\|^p+\|x,b\|^p)^{1/p}\\&\leq 2^{1/p}\max\{\|x,a\|,\|x,b\|\}=2^{1/p}\|x\|_{a,b,\infty},\end{aligned}$$

It means, that the norms $\|\cdot\|_{a,b,p}$ and $\|\cdot\|_{a,b,\infty}$ are equivalent.

Let $q\geq p\geq 1$. Further, using the already known inequality

$$(u^q+v^q)^{1/q}\leq(u^p+v^p)^{1/p},\ u,v\geq 0$$

we get

$$\|x\|_{a,b,q}=(\|x,a\|^q+\|x,b\|^q)^{1/q}\leq(\|x,a\|^p+\|x,b\|^p)^{1/p}=\|x\|_{a,b,p}. \quad (6)$$

On the other hand, without any general restrictions, we may take that for given $x\in L$ the inequality $\|x,b\|\leq\|x,a\|$ is satisfied. Then,

$$\begin{aligned}\|x\|_{a,b,p} &= (\|x,a\|^p + \|x,b\|^p)^{1/p} = \|x,a\|[1+(\tfrac{\|x,b\|}{\|x,a\|})^p]^{1/p} \\ &\le 2^{1/p}\|x,a\| \le 2^{1/p}\|x,a\|[1+(\tfrac{\|x,b\|}{\|x,a\|})^q]^{1/q} \\ &= 2^{1/p}(\|x,a\|^q + \|x,b\|^q)^{1/q} = 2^{1/p}\|x\|_{a,b,q}.\end{aligned} \tag{7}$$

Finally, the inequalities (6) and (7) implies that the norms $\|\cdot\|_{a,b,p}$ and $\|\cdot\|_{a,b,q}$ are equivalent. ■

Let $\{a,b\}$ be linearly independent set in 2-normed space L. Then, 2-norm induces family of norms $\{\|\cdot\|_{a,b,\infty},\ \|\cdot\|_{a,b,p},\ p\ge 1\}$. Furthermore, for $p\ge 1$ the norms are given by (4), and for $p=\infty$ the norm is given by (5). Now, let $\{a,b\}$ and $\{c,d\}$ be linearly independent sets. Let review the families of norms $\{\|\cdot\|_{a,b,\infty},\ \|\cdot\|_{a,b,p},\ p\ge 1\}$ and $\{\|\cdot\|_{c,d,\infty},\ \|\cdot\|_{c,d,p},\ p\ge 1\}$. Clearly, if L is a space with finite dimension, then each two norms of reviewed families are equivalent (theorem 2, [6], pp. 29).

But problems of equivalence between the norms,

1) $\|\cdot\|_{a,b,\infty}$ and $\|\cdot\|_{c,d,\infty}$,

2) $\|\cdot\|_{a,b,\infty}$ and $\|\cdot\|_{c,d,p}$, $p\ge 1$ and

3) $\|\cdot\|_{a,b,p}$ and $\|\cdot\|_{c,d,q}$, $p,q\ge 1$

and the conditions which must be satisfied if L be a space with not finite dimension are still opened.

Example 1. If $(L,(\cdot,\cdot))$ be a real pre-Hilbert space, then

$$(x,y\,|\,z) = \begin{vmatrix}(x,y) & (x,z)\\ (y,z) & (z,z)\end{vmatrix},\quad x,y,z\in L \tag{8}$$

defines a 2-inner product. It's obvious that (1) defines 2-norm on L, i.e.

$$\|x,y\| = \sqrt{\|x\|^2\|y\|^2-(x,y)^2}. \tag{9}$$

Further, if $\{a,b\}$ is a linearly independent subset of L, then

$$\|x\|_{a,b,p} = [(\|x\|^2\|a\|^2-(x,a)^2)^{p/2} + (\|x\|^2\|b\|^2-(x,b)^2)^{p/2}]^{1/p},\quad p\ge 1$$

$$\|x\|_{a,b,\infty} = \max\{\sqrt{\|x\|^2\|a\|^2-(x,a)^2},\ \sqrt{\|x\|^2\|b\|^2-(x,b)^2}\},$$

is a family of norms, which are generated by the prime norm $\|x\|=\sqrt{(x,x)}$, and for every $p\ge 1$

$$\|a\|_{a,b,p} = \|b\|_{a,b,p} = \|a\|_{a,b,\infty} = \|b\|_{a,b,\infty} = \sqrt{\|a\|^2\|b\|^2-(a,b)^2}.$$

Clearly, if L is a space with finite dimension, all these norms are equivalent to the prime norm. But, the following question is still opened: Is it true that for every vectors a and b the prime norm $\|\cdot\|$ is equivalent to the norms $\|\cdot\|_{a,b,p}$, $p\ge 1$ and $\|\cdot\|_{a,b,\infty}$ ■

III. SOME PROPERTIES INHERED FROM THE SPACE $(L,\|\cdot,\cdot\|)$ TO $(L,\|\cdot\|_{a,b,p})$, $p\ge 1$ TYPE OF SPACES

Theorem 4. If $(L,\|\cdot,\cdot\|)$ be 2-pre-Hilbert space, then for any linearly independent set $\{a,b\}$ the normed space $(L,\|\cdot\|_{a,b,2})$ be pre-Hilbert, and further more for each $x,y\in L$ is true that

$$(x,y)_{a,b} = (x,y\,|\,a) + (x,y\,|\,b). \tag{10}$$

Proof. Equalities (3) and (4) imply that for each $x,y\in L$

$$\begin{aligned}\|x+y\|_{a,b,2}^2 + \|x-y\|_{a,b,2}^2 &= \|x+y,a\|^2 + \|x+y,b\|^2 + \|x-y,a\|^2 + \|x-y,b\|^2 \\ &= 2(\|x,a\|^2+\|y,a\|^2) + 2(\|x,b\|^2+\|y,b\|^2) \\ &= 2(\|x,a\|^2+\|x,b\|^2) + 2(\|y,a\|^2+\|y,b\|^2) \\ &= 2(\|x\|_{a,b,2}^2+\|y\|_{a,b,2}^2),\end{aligned}$$

It means that in the space $(L,\|\cdot\|_{a,b,2})$ the parallelogram equality is succeeded. It implies that the mentioned space is pre-Hilbert space. Further,

$$(x,y)_{a,b}=\frac{\|x+y\|_{a,b,2}^2-\|x-y\|_{a,b,2}^2}{4}=\frac{\|x+y,a\|^2+\|x+y,b\|^2-\|x-y,a\|^2-\|x-y,b\|^2}{4}$$
$$=\frac{\|x+y,a\|^2-\|x-y,a\|^2}{4}+\frac{\|x+y,b\|^2-\|x-y,b\|^2}{4}=(x,y\,|\,a)+(x,y\,|\,b),$$

i.e. the equality (10) is true. ■

Remark 1. By Theorem 4 we proved that if $(L,\|\cdot,\cdot\|)$ be 2-pre-Hilbert space, the normed space $(L,\|\cdot\|_{a,b,2})$ is pre-Hilbert. Among each norms $\|\cdot\|_{a,b,p}$, $1\le p\le\infty$ on $\mathbf{R}^n$ get in Example 1 only the norm $\|\cdot\|_{a,b,2}$ is induced by inner product. Really, if

$$a=(1,1,0,...,0),\; b=(1,0,1,0,...,0),\; x=(0,1,0,...,0) \text{ and } y=(0,0,1,0,...,0),$$

for $p\ne 2, 1\le p<\infty$ we get

$\|x\|_{a,b,p}=(1+2^{p/2})^{1/p}$, $\|y\|_{a,b,p}=(2^{p/2}+1)^{1/p}$,

$\|x+y\|_{a,b,p}=2^{1/p}3^{1/2}$ и $\|x-y\|_{a,b,p}=2^{1/p}3^{1/2}$,

thus,

$$\|x+y\|_{a,b,p}^2+\|x-y\|_{a,b,p}^2=6\cdot 4^{1/p}\ne 4(1+2^{p/2})^{2/p}=2(\|x\|_{a,b,p}^2+\|y\|_{a,b,p}^2).$$

It means that the parallelogram equality is not satisfied. Further, for $p=\infty$ we get

$$\|x\|_{a,b,\infty}=\sqrt{2},\; \|y\|_{a,b,\infty}=\sqrt{2},\; \|x+y\|_{a,b,\infty}=\sqrt{3} \text{ и } \|x-y\|_{a,b,\infty}=\sqrt{3},$$

thus

$$\|x+y\|_{a,b,\infty}^2+\|x-y\|_{a,b,\infty}^2=6\ne 8=2(\|x\|_{a,b,\infty}^2+\|y\|_{a,b,\infty}^2),$$

It means that this is other case in which the parallelogram equality is not satisfied.

Remark 2. If $(L,(\cdot,\cdot))$ be a real pre-Hilbert space, then (8) defines 2-inner product. Further, if $\{a,b\}$ be a linearly independent set, then by theorem 4, equality (10) defines an inner product on L

$$(x,y)_{a,b}=(x,y\,|\,a)+(x,y\,|\,b)=(x,y)[\|a\|^2+\|b\|^2]-(x,a)(y,a)-(x,b)(y,b).$$

It means that using the prime inner product, we generate a family of inner products:

$$(\cdot,\cdot)_{a,b}, \text{ set } \{a,b\} \text{ is linearly independent on } L. \tag{11}$$

The real question is, either this family contains the prime inner product, i.e. are there exist linearly independent vectors $a,b\in L$ such that for every $x,y\in L$ is true that

$$(x,y)_{a,b}=(x,y). \tag{12}$$

But $(a,b)_{a,b}=0$. Thus, if exist linearly independent vectors $a,b\in L$ such that for every $x,y\in L$ (12) is hold, then $(a,b)=0$. Further, letting $x=y=a$ in equality (12) and considering $(a,b)=0$ and $\|a\|>0$, we get $\|b\|=1$. Analogously, we get $\|a\|=1$. Hence, the equality (12) is transformed as

$$(x,y)=(x,a)(y,a)+(x,b)(y,b). \tag{13}$$

Two cases are possible:

1. $\dim L=2$. Then, the set $\{a,b\}$ is orthonormed base on L. In fact, the equality (13) is a Parseval equality, and thus, the family (11) contains the prime inner product. The same inner product is get for each orthonormed base $\{a,b\}$ of L.

The last means if $\{a,b\}$ be orthonormed base of L then the prime norm $\|\cdot\|$ is identical to the norm $\|\cdot\|_{a,b,2}$.

2. $\dim L>2$. Then, by Gram-Schmidt Theorem for orthogonalization, exists $c\in L$ such that $(a,c)=(b,c)=0$ and $\|c\|=1$. Letting $x=y=c$ in the equality (13) we get

$$1=\|c\|^2=(c,a)^2+(c,b)^2=0,$$

and that is contradiction. The last implies the family (11) doesn't contain the prime inner product. It means, for $\dim L>2$, there is no any norm $\|\cdot\|_{a,b,p}$, $1\le p\le\infty$ which is identically to $\|\cdot\|$.

Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then, by lemma 2.1, [1], on $L\times L\times L$ exist the functional above

$$N_+(x,z)(y)=\lim_{t\to 0^+}\frac{\|x+ty,z\|-\|x,z\|}{t},\quad N_-(x,z)(y)=\lim_{t\to 0^-}\frac{\|x+ty,z\|-\|x,z\|}{t},$$

and are *called right-hand and left-hand Gateaux derivative*, respectively of a 2-norm $\|\cdot,\cdot\|$ at (x,z) in the direction y. Further, if $N_-(x,z)(y)=N_+(x,z)(y)$, then the 2-norm $\|\cdot,\cdot\|$ is said to be *Gateaux differentiable* at (x,z) in the direction y and is denoted by

$$N(x,z)(y) = \lim_{t\to 0} \frac{\|x+ty,z\|-\|x,z\|}{t}.$$

2-normed space $(L,\|\cdot,\cdot\|)$ is called to be *smooth* if for $x \neq 0$ and $z \notin V(x)$ the 2-norm $\|\cdot,\cdot\|$ is Gateaux differentiable at (x,z) in the direction y ([7]).

Theorem 5. If 2-normed space $(L,\|\cdot,\cdot\|)$ is smooth, then the normed space $(L,\|\cdot\|_{a,b,1})$ is smooth for each linearly independent set $\{a,b\}$.

Proof. Let 2-normed space $(L,\|\cdot,\cdot\|)$ is smooth and $\{a,b\}$ is linearly independent set. Then,

$$N_-(x,a)(y) = N_+(x,a)(y) \text{ и } N_-(x,b)(y) = N_+(x,b)(y)$$

thus,

$$\tau_+(x,y) = \lim_{t\to 0^+} \frac{\|x+ty\|_{a,b,1}-\|x\|_{a,b,1}}{t} = \lim_{t\to 0^+} \frac{\|x+ty,a\|+\|x+ty,b\|-\|x,a\|-\|x,b\|}{2}$$

$$= \lim_{t\to 0^+} \frac{\|x+ty,a\|-\|x,a\|}{2} + \lim_{t\to 0^+} \frac{\|x+ty,b\|-\|x,b\|}{2}$$

$$= \lim_{t\to 0^-} \frac{\|x+ty,a\|-\|x,a\|}{2} + \lim_{t\to 0^-} \frac{\|x+ty,b\|-\|x,b\|}{2})$$

$$= \lim_{t\to 0^-} \frac{\|x+ty,a\|+\|x+ty,b\|-\|x,a\|-\|x,b\|}{2} = \lim_{t\to 0^-} \frac{\|x+ty\|_{a,b,1}-\|x\|_{a,b,1}}{t} = \tau_-(x,y)$$

It means that the normed space $(L,\|\cdot\|_{a,b,1})$ is smooth. ■

The terms convergent sequence and Cauchy sequence in 2-normed space are given by A. White. The sequence $\{x_n\}_{n=1}^{\infty}$ in 2-normed space is called to be *convergent* if there exists $x \in L$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$, for every $y \in L$. The vector $x \in L$ is called to be bound of the sequence $\{x_n\}_{n=1}^{\infty}$ and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$, ([8]). The sequence $\{x_n\}_{n=1}^{\infty}$ in 2-normed space L is called to be *Cauchy* if for every $y \in L$, $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$, ([9]).

Theorem 6. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in 2-normed space $(L,\|\cdot,\cdot\|)$ and $\{a,b\}$ be linearly independent set in L.

a) If the sequence $\{x_n\}_{n=1}^{\infty}$ be Cauchy sequence in $(L,\|\cdot,\cdot\|)$, then that sequence is Cauchy sequence in $(L,\|\cdot\|_{a,b,p})$, $p \geq 1$ and in $(L,\|\cdot\|_{a,b,\infty})$, too.

b) If the sequence $\{x_n\}_{n=1}^{\infty}$ be convergent sequence in $(L,\|\cdot,\cdot\|)$, then that sequence is convergent sequence in $(L,\|\cdot\|_{a,b,p})$, $p \geq 1$ and in $(L,\|\cdot\|_{a,b,\infty})$, too.

Proof. a) Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy sequence in $(L,\|\cdot,\cdot\|)$. Then,

$$\lim_{m,n\to\infty} \|x_n - x_m, a\| = 0 \text{ and } \lim_{m,n\to\infty} \|x_n - x_m, b\| = 0,$$

So, for each $p \geq 1$,

$$\lim_{m,n\to\infty} \|x_n - x_m\|_{a,b,p} = \lim_{m,n\to\infty} (\|x_n - x_m, a\|^p + \|x_n - x_m, b\|^p)^{1/p} = 0 \text{ and}$$

$$\lim_{m,n\to\infty} \|x_n - x_m\|_{a,b,\infty} = \lim_{m,n\to\infty} \max\{\|x_n - x_m, a\|, \|x_n - x_m, b\|\} = 0,$$

i.e. $\{x_n\}_{n=1}^{\infty}$ be Cauchy sequence in $(L,\|\cdot\|_{a,b,p})$, $p \geq 1$ and $(L,\|\cdot\|_{a,b,\infty})$.

b) Let $\{x_n\}_{n=1}^{\infty}$ be convergent sequence in $(L,\|\cdot,\cdot\|)$. Then, there is $x \in L$ such that,

$$\lim_{n\to\infty} \|x_n - x, a\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - x, b\| = 0,$$

So, for each $p \geq 1$

$$\lim_{n\to\infty} \|x_n - x\|_{a,b,p} = \lim_{n\to\infty} (\|x_n - x, a\|^p + \|x_n - x, b\|^p)^{1/p} = 0 \text{ and}$$

$$\lim_{n\to\infty} \|x_n - x\|_{a,b,\infty} = \lim_{n\to\infty} \max\{\|x_n - x, a\|, \|x_n - x, b\|\} = 0,$$

i.e. $\{x_n\}_{n=1}^{\infty}$ be convergent sequence in $(L,\|\cdot\|_{a,b,p})$, $p\geq 1$ and $(L,\|\cdot\|_{a,b,\infty})$. ■

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