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The Expected Number of Real Zeros of Random Polynomial

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Abstract:- In this paper we have estimated the number of real zeros of $Q_n(Z) = \sum_{j=0}^{n} X_j z^j, z \in Z$ which is a random Gaussian polynomial satisfying the normal distribution with mean zero and variance one i.e. $E(X_j) = 0$ and $E(X_j)^2 = 1$ for $j \geq 0$. Much research works has been done on the same polynomial with different co-efficients satisfying the above condition and found that the expected number of zeros is approximated to $\left(2/\pi\right)\log n$ as $n \to \infty$ in the interval $(-\infty, \infty)$. Our present work is to estimate the number of zeros in the interval [0, 1] and found that the expected number of real zeros of the above polynomial under same conditions is $ENn[0,1] \sim \left(1/2\pi\right)\log n$ as $n \to \infty$. Our result gives better approximation as compared to results given by Yoshihara [6]

Keywords: - *Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots.*

INTRODUCTION

Let $X_0, X_1 \ldots\ldots X_n$ be a stationary Gaussian process satisfying $E(X_j) = 0$ and $E(X_j)^2 = 1$ for $j \geq 0$ as sufficient conditions. Then the expected number of real zeros of the above polynomial $Q_n(z) = \sum_{j=0}^{n} X_j z^j$ is $\frac{1}{2\pi}(\log n)$ as n tends to infinity. The expected number of zeros of a random trigonometric polynomial has been studied by Dunnage[1] and estimated the number of zeros in the interval [0,1] which is approximated to $(1/2\pi)\log n$ where $n \to \infty$ but in the same problem we consider a polynomial which is piece wise continuous and differentiable in the same interval and used normal distribution with mean zero and variance one and applied Gaussian process. Ibragimov and Maslova [2] had worked with same mean and variance under different conditions and had got the expected number of zeros in the same interval is approximated to the result of Dunnage [1].

Let us consider the Gaussian polynomial

$$Q_n(z) = \sum_{j=0}^{n} X_j z^j \tag{1}$$

Which is a piece-wise continuous and differentiable polynomial within a closed interval [0, 1] and satisfying same conditions. Suppose that there is an interval [-b,b] , $0<b\leq \pi$, where $f(\theta)$ is uniformly approximated by the partial sums, $S_n f(\theta)$ of its Fourier series development, and $0 \leq m \leq f(\theta) \leq M \leq \infty$, $\theta \in [-\pi, \pi]$ where m and M are the lower and upper bounds of $f(\theta)$. Then the expected number of real zeros of the above polynomial is $ENn\{[0,1]\} \sim \left(\frac{1}{2\pi}\right)\log n$ as $n \to \infty$.

Let the number of zeros of the above polynomial is denoted by $EN_{n.}$ .The main aim of our work is to estimate $ENn\{[0,1]\}$.

Now we have partitioning the interval [0, 1] into three different sub intervals namely I_n^1, I_n^2 & I_n^3 the details of partitions are given bellow,

$$I_n^1 = \left[0, (1-\frac{1}{\sqrt{\log n}})\right]$$

$$I_n^2 = \left[(1-\frac{1}{\sqrt{\log n}}), (1-\frac{\log\log n}{n})\right]$$

$$I_n^3 = \left[(1-\frac{\log\log n}{n}), 1\right]$$

For $n \geq 3$,

Let N_n (a,b) be the number of sign changes of a piece-wise linear approximation to $Q_n(x)$.

First we estimate $EN_n(I_n^2)$. Then at last section we have approximated $EN_n(I_n^1)$ and $EN_n(I_n^3)$ and found that the expected number of zeros of both the intervals are equivalent to σ(logn) as n tends to infinity

i.e. $EN_n(I_n^1) + EN_n(I_3^3) = \sigma(\log n)$

COVARIANCE ESTIMATES:

Let $k_n(x,y)=E\{Q_n(x)Q_n(y)\}$ and $r_n(x,y)=Cor\{Q_n(x), Q_n(y)\}$. We estimate $k_n(x,y)$ and $r_n(x,y)$ for x,y satisfying certain conditions. For $x \in I_n^2$ we derive upper and lower bounds for $k_n(x,x)$.

Let $T_n(\theta)$ be a trigonometric polynomial of order n with real coefficients. Suppose we define

$$T_n(\theta) = \sum_{v=-n}^{n} c_v e^{iv\theta},\ \theta \in [-\pi, \pi] \text{ and } c_v \in R \tag{2}$$

$$G(T_n, x) = 2\pi\left(\sum_{v=0}^{n} c_v x^v - c_0/2\right) \tag{3}$$

Then

$$k^a(T_n, x, y) = \frac{G(T_n, x) + G(T_n, y)}{1-xy} \qquad x,y \in (0,1]$$

Taking $0 \leq d \leq r'(x, y) \leq \infty$

$$r'(x,y) = \frac{(1-x^2)^{1/2}(1-y^2)^{1/2}}{1-xy} \qquad x,y \in (0,1]$$

We estimate $k_n(x,y)$ satisfying the conditions

$0 < d < \infty$ And $x, y \in I_n^2$

We know from co-variance estimates

$$\sup_{\substack{x,y\in I_n^2 \\ r'(x,y)\geq d>0}} \left| r_n(x,y) - r'(x,y) \right| \to 0 \quad \text{as } n \to \infty$$

For $x, y \in I_n^1 \cup I_n^2$ we have

THEOREM -1 Suppose that $f(\theta)$ can be uniformly approximated by the partial sums of its Fourier series development $f(\theta) \leq A < \infty, \theta \in [-\pi, \pi]$ & f (0)>0. If there is a constant α and an integer N_0 such that $G(S_n f, x) \geq \alpha > 0$ for $n \geq N_0$, $x \in [0,1]$ Applying co-variance estimates to find mean and variance of the series $\sum_{j=0}^{m} a_j X_j$ we have

$$E\left(\sum_{j=0}^{m} a_j X_j\right)^2 \leq 2\pi A \sum_{j=0}^{m} a_j^2 \quad \text{for } m \geq 0 \text{ and } k_n(x,y) = \int_{-\pi}^{\pi} \left(\sum_{v=0}^{n} x^v e^{-iv\theta}\right)\left(\sum_{v=0}^{n} y^v e^{iv\theta}\right) f(\theta) d\theta$$

$$Let \ k(x, y) = E\{Q(x)Q(y)\} = \lim_{n\to\infty} k_n(x,y)$$

$$= \int_{-\pi}^{\pi} \left(1 - xe^{-i\theta}\right)^{-1} \left(1 - ye^{i\theta}\right)^{-1} f(\theta) d\theta \tag{4}$$

$$\text{Let } T_m(\theta) = \sum_{v=-m}^{m} c_v e^{iv\theta} \tag{5}$$

We have by Cauchy's residue theorem that $\int_{-\pi}^{\pi} e^{iv\theta}\left(1 - xe^{-i\theta}\right)^{-1}\left(1 - ye^{i\theta}\right)^{-1} d\theta = 2\pi x^v (1 - xy)^{-1}$

And $\int_{-\pi}^{\pi} e^{-iv\theta}\left(1 - xe^{-i\theta}\right)^{-1}\left(1 - ye^{i\theta}\right)^{-1} d\theta = 2\pi y^v (1 - xy)^{-1}$

$$\left| h(x,y) - \int_{-b}^{b} \left(1 - xe^{-i\theta}\right)^{-1}\left(1 - ye^{i\theta}\right)^{-1} g(\theta)\, d\theta \right| \leq C \tag{6}$$

Where C is a constant depending on B and b.
Taking m=n in equation (5) we have *m*=n and $T_n(\theta)=S_n f(\theta)$, where $S_n f(\theta)$ is the nth partial sum of the Fourier series development of f(θ). Now for $x, y \in [0,1]$ we have

$$\left| k_n(x,y) - k^a(S_n f, x, y) \right| \leq J_0 \tag{7}$$

Where $J_0 = J_1 + J_2 + J_3 + J_4$ and

$$J_1 = \left| k^a(S_nf,x,y) - \int_{-b}^{b} \left(1-xe^{-i\theta}\right)^{-1}\left(1-ye^{i\theta}\right)^{-1} S_nf(\theta)d\theta \right| \quad (8)$$

$$J_2 = \left| k(x,y) - \int_{-b}^{b} \left(1-xe^{-i\theta}\right)^{-1}\left(1-ye^{i\theta}\right)^{-1} f(\theta)d\theta \right| \quad (9)$$

$$J_3 = \left| \int_{-b}^{b} \left(1-xe^{-i\theta}\right)^{-1}\left(1-ye^{i\theta}\right)^{-1} (S_nf(\theta) - f(\theta)d\theta \right| \quad (10)$$

$$J_4 = \left| k_n(x,y) - k(x,y) \right| \quad (11)$$

From the conditions of Zygmund [7] we found that there is a constant B not depending on n such that

$$\int_{-\pi}^{\pi} |S_nf(\theta)|d\theta \le B < \infty, \ for\ all\ n \ge 0. \ \&\ g(\theta) = S_nf(\theta) \text{ gives } \quad J_1 \le C < \infty$$

Where C depends on b and B & g(θ)= f(θ) gives

Let $\quad a(n) = \sup_{\theta\in[-b,b]} \left|S_nf(\theta) - f(\theta)\right|$

By Cauchy's inequality we have

$$J_3 \le \frac{2\pi a(n)}{(1-x^2)^{1/2}(1-y^2)^{1/2}} \qquad \left|k_n(x,y) - k^a(S_nf,x,y)\right|$$

$$\le C\left(1 + \frac{a(n) + x^{n+1} + y^{n+1} + (xy)^{n+1}}{\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2}}\right)$$

For $x, y \in [0,1]$ and where C is a constant depending upon b, A and B So

$$\left|k_n(x,y) - k^a(S_nf,x,y)\right| \le \frac{w(n)}{\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2}} \quad (12)$$

Where $w(n) \to 0 \quad as \quad n \to \infty$ For $x, y \in I_n^1 \cup I_n^2$

Now $\quad G(S_nf,x) = \frac{1}{2} + \sum_{v=1}^{n} r_v x^v \quad (13)$

We show that there is a constant α>0 and an integer N_0 such that

$G(S_nf,x) \ge \alpha > 0 \quad when \quad x \in I_n^2 \quad , n \ge N_0$ From Abel's Theorem and Titchmarsh[4] conditions we see that $G(S_nf,x)$ is uniformly convergent for $x \in [0,1]$ and

$$\lim_{x\to 1^-} G(S_\infty f,x) = \frac{1}{2} + \sum_{v=1}^{n} r_v = \pi f(0) \quad (14)$$

PRROF OF THEOREM -2 Using the two conditions $f(\theta) \le A < \infty, \theta \in [-\pi,\pi]$ and f(0)>0 of Theorem-1 and applying the uniform convergence of the series within a certain interval $S_nf(\theta)\ \theta \in [-\pi,\pi]$ and $\pi S_nf(0) = G(S_nf,0)$ by adopting similar procedure as in the proof of Theorem -1 we see that $J_1=J_2=0$ holds for $x \in I_n^1 \cup I_n^2$. For some μ we construct an interval $\theta \in [-\pi, \pi]$ and $f(\theta) \geq \mu > 0$. Such an interval exists as $f(\theta)$ is continuous at θ=0 and

$f(0)$ >0. We consider the case when $x \in \left(1-(\log n)^{-1/2}, 1\right)$ and x=1 separately.
The following inequality holds for $x, y \in [0,1)$

$\sup_{0<b\leq|\theta|\leq\pi} \left|1-xe^{-i\theta}\right|^{-1} \leq \left(1-\cos^2 b\right)^{-1/2}, \pi/2 \geq b > 0$ $\sup_{0<b\leq|\theta|\leq\pi} \left|1-xe^{-i\theta}\right|^{-1} \leq 1$ for $\pi \geq b > \pi/2$ We have

$$\sup_{0<b\leq|\theta|\leq\pi} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right| \quad \text{for } x \in [0,1)$$

Substitution in $k_n(x,x)$ with $\left|1-xe^{-iv\theta}\right|^2$ replaced by $\left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2$

We got

$$\left|k_n(x,x) - \int_{-b}^{b} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 f(\theta)d\theta\right| \leq C \quad x \in [0,1) \tag{15}$$

Substitute $f(\theta) = 1/2\pi$ in (15) we have

$$\left|\int_{-\pi}^{\pi} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 d\theta - \int_{-b}^{b} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 d\theta\right| \leq C \tag{16}$$

Where C depends only on b and A.

By simple calculation we have

$$\int_{-\pi}^{\pi} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 d\theta = 2\pi \sum_{v=0}^{n} x^{2v}, \quad x \in [0,1] \tag{17}$$

$$\int_{-b}^{b} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 d\theta \sim 2\pi \sum_{v=0}^{n} x^{2v}, n \to \infty \tag{18}$$

and $x \in \left[1-(\log n)^{-1/2}, 1\right]$.

From our construction of [-b,b] we have

$$\int_{-b}^{b} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 f(\theta)d\theta \geq \mu \int_{-b}^{b} \left|\sum_{v=0}^{n} x^v e^{-iv\theta}\right|^2 d\theta \tag{19}$$

So the desired result follows for $x \in \left[1-(\log n)^{-1/2}, 1\right)$.
When x=1 we have

$$k_n(1,1) = n+1+2\sum_{j=1}^{n}(n-j+1)r_j = 2\pi(n+1)\sigma_n f(0) \tag{20}$$

Where $\sigma_n f(\theta)$ is the nth Cesaro sum associated with $f(\theta)$. As $f(\theta)$ is continuous at $\theta=0$ we have $\sigma_n f(0) \to f(0)$ as $n \to \infty$. Hence we

have $$E\left(\sum_{j=0}^{m} a_j x_j\right)^2 = \int_{-\pi}^{\pi}\left(\sum_{v=0}^{m} a_v e^{-iv\theta}\right)\left(\sum_{v=0}^{m} a_v e^{iv\theta}\right) f(\theta)d\theta$$

given that $f(\theta) \le A < \infty$. Then by using Cauchy's inequality we have

$$E\left(\sum_{j=0}^{m} a_j x_j\right)^2 \le A\int_{-\pi}^{\pi}\left|\sum_{v=0}^{m} a_v e^{-iv\theta}\right|^2 d\theta \tag{21}$$

By simple calculation we have

$$\int_{-\pi}^{\pi}\left|\sum_{v=0}^{m} a_v e^{-iv\theta}\right|^2 d\theta = 2\pi\sum_{j=0}^{m} a_j{}^2 \tag{22}$$

GENERAL APPROXIMATION FORMULA : - For $x \in [a,b]$ we approximate $Q_n(x)$ by a process which linearly interpolates between $Q_n(a)$ and $Q_n(b)$. It is convenient to count the sign changes of this process in [a,b] by

$$N_n(a,b) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\operatorname{sgn}\{Q_n(a)Q_n(b)\} \tag{23}$$

Where $$\operatorname{sgn}\{x\} = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

The results of this section depend on an upper bound for the number of zeros of $Q_n(x)$ in an interval [a,b]. Define the event

$$U_k = Q_n(x) \text{ has k more zeros in } [a,b] \tag{24}$$

For any interval $[a,b] \subseteq [0,1]$ let

$\gamma = (b-a)(1-b)^{-1}, \quad b \in [0, 1-(n+1)^{-1}]$ and

$\gamma = (n+1)(b-a), \quad b \in [1-(n+1)^{-1}, 1]$

LEMMA -1 let $\gamma < 2^{-30}$ *and*

(i) $f(\theta)$ is continuous at $\theta=0$

(ii) $f(0) > 0$

(iii) $f(\theta) \le A < \infty, \theta \in [-\pi, \pi]$

Then there is an integer N_1 and an absolute constant C such that

$$P(U_k) < C\gamma^{3k/5} \quad n \ge N_1 \quad k \ge 0$$

COROLLARY-1 Under the conditions of Lemma-1 we have

$$|EN_n(a,b) - EN_n([a,b])| < C\gamma^{6/5}, \quad n \ge N_1$$

Where C is an absolute constant.

Now we have to estimate the number of zeros in the three sub intervals.

(A). ESTIMATION OF NUMBER OF ZEROS IN THE INTERVAL:- I_n^1

LEMMA -2 Let X_0, X_1……X_n be a set of n points satisfying $E(X_j)=0$ and $E(X_j)^2=1$ for $j \geq 0$.

(i)Let us consider a function $f(\theta)$ which is uniformly approximated by the partial sums, $S_n f(\theta)$ in the interval [-π,π]

(ii) In the interval $0 \leq m \leq f(\theta) \leq M \leq \infty$ and $\theta \in [-\pi, \pi]$ where m and M are the lower and upper bounds of $f(\theta)$. Then expected number of real zeros in the interval

$$I_n^1 = \left[0, (1-\frac{1}{\sqrt{\log n}})\right] \text{ is } ENn(I_n^1) \leq (C^{1/2}/2\pi)\log\log n \quad n \geq 2 \tag{25}$$

Where C is a finite constant

PROOF: - From the Kac-Rice [3] formula and using the postulates of Shankar [4] which states that

$$EN_n([a,b]) = (\frac{1}{\pi})\int_a^b C_n^{1/2} B_n^{-1/2}(1-R_n^2)^2 dx \tag{26}$$

$$B_n = E(Q_n(x))^2, \ C_n = E(Q_n'(x))^2 \tag{27}$$

$$R_n = Cor(Q_n(x), Q_n'(x)) \tag{28}$$

Where $\frac{d}{dx}Q_n(x) = Q_n'(x)$

Now we have to find an upper bound for C_n/B_n with $x \in I_n^1$

We represent $B_n=k_n(x,x)$ by Noting that $f(\theta) \geq m > 0, \theta \in [-\pi,\pi]$ and applying Lemma-1 gives $B_n \geq 2\pi m \sum_{v=0}^{n} x^{2v}$ for $x \in [1,0]$ From Lemma-2 we have

$C_n \leq 2\pi A\left(\sum_{v=1}^{n} v^2 x^{2(v-1)} a_v e^{iv\theta}\right)$ summing $\sum_{v=1}^{n} x^{2v}$ and using that $\sum_{v=1}^{n} v^2 x^{2(v-1)} \leq 2(1-x^2)^{-3}$

$$x \in [0,1) \tag{29}$$

$$C_n/B_n \leq (2A/m)(1-x^{2(n+1)})^{-1}(1-x^2)^{-2} \tag{30}$$

$$x^{2(n+1)} < 1, \ x \in I_n^1 \text{ for } n \geq 2$$

and $\sup_{x \in I_n^1} x^{2(n+1)} \to 0, x \in I_n^1$ as $n \to \infty$ We deduce that $(1-x^{2(n+1)})^{-1}$ is bounded above by a constant. So

$$C_n/B_n \leq (1-x)^{-2} \qquad n \geq 2 \text{ and } x \in I_n^1 \tag{31}$$

Substituting the value of C_n/B_n and using the relation $(1-R_n^2) \leq 1$ gives the expected number of zeros in the interval I_n^1 is $ENn(I_n^1) \leq (C^{1/2}/2\pi)\log\log n$ for $n \geq 2$ (32)

Hence the theorem is proved.

LEMMA-3 Let X_0, X_1……X_n be a set of n stationary points satisfying $E(X_j)=0$ and $E(X_j)^2=1$. for $j \geq 0$ Suppose that (i) there is an interval $[-\pi,\pi]$ and there is a constant α such that $\frac{1}{2}+\sum_{j=1}^{n} r_j x^j \geq \alpha > 0 \quad x \in [0,1] \quad n \geq N_0$, for some integer N_0 there is a constant C such that

$$EN_n\left(I_n^1\right) \leq \left(\frac{C^{1/2}}{2\pi}\right) \log \operatorname{logn} \text{ for } n \geq N_5$$

PROOF: - To find an upper bound for C_n/B_n, x∈ I_n^1 and applying Lemma-1 and Lemma-2 we have $G(S_n f, x) = \frac{1}{2}+\sum_{j=1}^{n} r_j x^j \geq \alpha > 0$ for $n \geq N_0$ (33)

Let N_n be any integer

So the conditions of Theorem-2 are satisfied. From Theorem-2 and any $C \in (0,1)$ we have an integer N_4 such that

$B_n = k_n(x,x) > CK^a(S_n f, x, x), \quad n \geq N_4 \;\; \& \;\; x \in I_n^1$ (34)

Where C being any constant and $C \in (0,1)$

From LEMMA-2 we have

$$K^a(S_n f, x, x) = 2\frac{G(S_n f, x)}{1-x^2}$$

But $G(S_n f, x) \geq \alpha > 0$ *for* $n \geq N_0$. So

$B_n > \frac{2\alpha C}{1-x^2}, n \geq N_5 = \max(N_0, N_4)$ (35)

In the same way as in Lemma- 2 we have

$B_n / C_n \leq (1-x)^{-2}$, $n \geq N_5 = \max(N_0, N_4)$. Then the expected number of zeros in the interval I_n^3 is

$EN_n\left(I_n^1\right) \leq \left(\frac{C^{1/2}}{2\pi}\right) \log \operatorname{logn}$ for $n \geq N_5$ (36)

Hence the theorem is proved.

LEMMA-4 Let $(Z_j, j \geq 0)$ be a stationary sequence of uniformly mixing random variables with zero mean and

(i) $E|Z_j|^{2+\delta} < \infty$ for $\delta > 0$

(ii). $E\left(\sum_{j=0}^{n} Z_j\right)^2 \to \infty$ as $n \to \infty$

Then there exists a constant C such that

$$E\left|\sum_{j=0}^{n} Z_j\right|^{2+\delta} \leq C\left|E\left(\sum_{j=0}^{n} Z_j\right)^2\right|^{1+\delta/2}$$

THEOREM- 2 Let $X_0, X_1 \ldots\ldots X_n$ be a set of n stationary real-valued uniformly mixing Gaussian process with $E(X_j) = 0$ and $E(X_j)^2 = 1$ for $j \geq 0$ satisfying (i) $\sum_{k=1}^{\infty} \phi^{1/2}(k) < \infty$ and (ii) f(Θ) >0 and also satisfied by the sequence $\{(-1)^j X_j, j \geq 0\}$.

We have

$$EN_n\left(I_n^1\right) \le C(\log n)^{1/2} \log \log n$$

Where C is a constant.

PROOF: Choose C such that

$P(|X_0| \ge c) = q < 1$ and Choose the events $B_0 = (|X_0| < c)$ and

$B_k = (|X_0| < c.....|X_{k-1}| < c, |X_k| \ge c)$ and $k = 1,....n.$

$$B = (|X_0| < c......|X_n| < c)$$

Let $0 < r < R$.

Using the argument of Ibragimov and Maslova[2]

We obtain.

$$EN_n([-r,r]) \le \sum_{k=1}^{n} kP(B_k) + nP(B) + (\log R/r)^{-1} \sum_{k=0}^{n} \int_{B_k} H_s dP \text{ Where}$$

$$H_s = \log\left(\sup_{0 \in [-\pi,\pi]} (k!c)^{-1} \left| Q_n^{(k)}(\mathrm{R}_e^{i\theta}) \right| \right) \qquad (37)$$

Here $Q_n^{(k)}(x)$ is the k^{th} derivative of $Q_n(x)$ with respect to x.

We estimate $P(B_k)$ and $P(B)$

Define a sequence of random variables

$(Z_k, k=0,.n)$ by

$$Z_k = \begin{cases} 1-q & |X_k| \ge c \\ -q & |X_k| < c \end{cases}$$

Now for k=1…..n we have

$$B_k = \left(\left(\sum_{j=0}^{k-1} Z_j = -kq \right) \cap (Z_k = 1-q) \right) \text{and}$$

$$B_0 = (Z_0 = 1-q)$$

We show that the conditions of Lemma-4 are satisfied by $(Z_k, k \ge 0)$ with $\delta = 4$ Clearly $(Z_k, k \ge 0)$ is a stationary sequence of uniformly random variables with mixing coefficient $\phi(j)$.

Using Ibragimov [2] and $\displaystyle\sum_{j=1}^{\infty} \phi^{1/2}(j) < \infty$ (38)

gives $\displaystyle E\left(\sum_{j=0}^{n-1} Z_j \right)^2 \sim n \quad as \quad n \to \infty$ (39)

So condition (i) & (ii) is satisfied. Using, Markov's inequality and Lemma-4 with $\delta = 4$ gives

$$P(B_k) \le P\left(\left| \sum_{j=0}^{n} Z_j \right| \ge kq/2 \right) < C/k^3 \; for \quad C < \infty \quad and \quad k \ge 1 \text{And} \sum_{k=1}^{\infty} kP(B_k) < \infty$$

In the same way

$$P(B) \le P\left(\left|\sum_{j=0}^{n} Z_j\right| \ge q(n+1)/2\right) < C/n^3, C < \infty \qquad (40)$$

And $nP(B) \to 0,\ n \to \infty$

We estimate the final term in using the method of Ibragimov and Maslova [2]

$$\int_{B_k} H_s dP \le P(B_k)\log w_k + P(B_k)\log T + C(1+i_0)\exp(-i_0) \qquad (41)$$

Where C is a constant, $i_0 = \ln(T)$ and T>0 Then

$W_k = E\left(\sum_{j=k}^{n} \frac{j(j-1)....(j-k+1)R^{j-k}}{k!c}|X_j|\right)$ Taking T to be the following function of k

$$T = \begin{cases} 1, & k = 0 \\ k^{1+\varepsilon}, & k \ge 1 \end{cases} \qquad \text{for} \quad \varepsilon > 0$$

using Kac-rice formula and noting that

$W_k < C(1-R)^{-k-1}c^{-1} \qquad 0 \le R < 1$ we have

$$\sum_{k=0}^{n}\int_{B_k} H_s dP \le \left(\log\frac{C}{1-R}\right)\left(\sum_{k=0}^{n}(k+1)P(B_k)\right) + (1+\varepsilon)\sum_{k=0}^{n}(\log k)P(B_k)$$
$$+ D\sum_{k=1}^{n}\frac{1+(1+\varepsilon)\log k}{k^{1+\varepsilon}} - \log c$$

On Substituting r = 1- $(\log n)^{-1/2}$ and R=1-1/2 $(\log n)^{-1/2}$ in the above equation we get the expected number of real zeros in the interval I_n^1 is $(C^{1/2}/2\pi)\log\log n$ for $n \ge 2$ where C and D are constants

(B) ESTIMATION OF NUMBER OF ZEROS IN THE INTERVAL:- $\left(I_n^2\right)$

To find out the expected number of zeros in the interval

number $I_n^2 = \left[(1-\frac{1}{\sqrt{\log n}}), (1-\frac{\log\log n}{n})\right]$

Let the expected number of zeros of the above interval is denoted by $EN_n\left(I_n^2\right)$

LEMMA -5 Suppose that $0 < b \le \pi$ If

(i) $f(\theta)$ can be uniformly approximated in [-b,b] by the partial sums of its Fourier series development, $S_n f(\theta)$

(ii) *f(0)>0*

(iii) $f(\theta) \le A < \infty, \theta \in [-\pi, \pi]$

Then $EN_n\left(I_n^2\right) \sim (1/2\pi)\log n$ *as* $n \to \infty$

(C) ESTIMATION OF NUMBER OF ZEROS IN THE INTERVAL (I_n^3):-

LEMMA -6 If

(i) $f(\theta)$ is continuous at $\theta = 0$

(ii) $f(\theta) > 0$

(iii) $f(\theta) \le A < \infty, \theta \in [-\pi, \pi]$

Then there is a constant and an integer N_6 such that

$$EN_n\left(I_n^3\right) < C(\log\log n)^{7/5}, \qquad n \ge N_6$$

THEOREM -3 Let $X_0, X_1 \ldots\ldots X_n$ be a set of n points satisfying $E(X_j) = 0$ and $E(X_j)^2 = 1$. For $j \ge 0$. Suppose that there is an interval [-b, b], $0<b\le \pi$, where $f(\theta)$ is continuous at $\theta=0$, $\theta \in [-\pi, \pi]$, Then expected number of real zeros in the interval $I_n^3 = \left[(1 - \frac{\log\log n}{n}), 1\right]$ is

$$EN_n\left(I_n^3\right) < C(\log\log n)^{7/5} \qquad n \ge N_6$$

Where C is a finite constant

PROOF : - Now we have to find an upper bound for C_n/B_n with $x \in I_n^3$ using Kac-Rice formula $C_n/B_n \le (2A/m)(1 - x^{2(n+1)})^{-1}(1 - x^2)^{-2}$ So

$$C_n/B_n \le (1 - x)^{-2} \qquad n \ge 2 \text{ and } x \in I_n^3$$

Substituting for C_n/B_n in the above equation and n taking the help of the inequality $(1 - R_n^2) \le 1$ the expected number of zeros in the interval I_n^3 is

$$EN_n\left(I_n^3\right) < C(\log\log n)^{7/5} \qquad n \ge N_6 \ .$$

Hence the theorem is proved.

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