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Comparison Of Some Numerical Methods For The Solution Of First And Second Orders Linear Integro Differential Equations.

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Abstract: - This paper deals with the comparison of some numerical methods for the solutions of first and second orders linear integro differential equations. Two numerical methods employed are Standard and Perturbed Collocation using, in each case, power series and canonical polynomials as our basis functions. The results obtained for some examples considered show that the perturbed Collocation method by Canonical Polynomials proved superior over the Perturbed Collocation method by power series and the Standard Collocation method by power series and canonical polynomials respectively. Three examples are considered to illustrate the methods.

Keywords: - *Integro-Differential Equations, Standard and Perturbed Collocation, Power series Canonical Polynomials*

I. INTRODUCTION

Integro differential equation is an important aspect of modern mathematics and occurs frequently in many applied fields of study which include Chemistry, Physics, Engineering , Mechanics, Astronomy, Economics, Electro – Statics and Potential.

In recent years, there has been growing interest in the mathematical formulation of several risk phenomena and models. It is found that most of the models if not all, have always assumed integral or integro differential equations. As reported in literature, integro differential equations are very difficult to solve analytically (See [1]) and so numerical methods are required.

Several research works have been carried out in this area in recent years. Among the popular methods used by most numerical analyst are wavelet on bounded interval [2] , semiorthogonal Spline Wavelets [3], Orthogonal Wavelets [4], Wavelet-Galerkin Method [5] and Multi-Wavelet Direct Method [6] . Other methods include Quadrature Difference Method [7] , Adomain Decomposition Method [8] , Homototpy Analysis Method [9], Compact Finite Difference Method [10], Generalised Minimal Residual [11] and Variational Iteration Method [12].

Without loss of generality ,we consider the general second order linear integro-differential equation defined as:

$$P_o\, y(x) + P_1\, y'(x) + P_2\, y''(x) + \int_a^b k(x,t)\, y(t)\, dt = f(x) \qquad (1)$$

With the boundary conditions

$$y(a) + y'(a) = A \qquad (2)$$

And,

$$y(b) + y'(b) = A \qquad (3)$$

Where P_0 , P_1 , P_2 are constants, $k(x,t)$ and $f(x)$ are given smooth functions and $y(x)$ is to be determined.

Remark: In case of first- order Integro –Differential Equation considered, P_2 in equation (1) is set to zero with initial condition given as

$$y(a) = A \qquad (3a)$$

II. METHODOLOGY AND TECHNIQUES

In this section, we discussed the numerical methods mentioned above based on power series and canonical polynomials as the basis function for the solution of equations (1)—(3)

III. STANDARD COLLOCATION METHOD BY POWER SERIES (SCMPS)

We used this method to solve equations (1)—(3) by assuming power series approximation of the form:

$$y_N(x) = \sum_{r=0}^{N} a_r x^r \tag{4}$$

Where , a_r $(r \geq 0)$ are the unknown constants to be determined. Thus, equation (4) is substituted into equations (1), (2) and (3), we obtained

$$P_o y_N(x) + P_1 y'_N(x) + P_2 y_N''(x) + \int_a^b k(x,t) y_N(t) dt = f(x) \tag{5}$$

together with the boundary conditions

$$y_N(a) + y'_N(a) = A \tag{6}$$

and,

$$y_N(b) + y'_N(b) = A \tag{7}$$

Equation (5) is re-written as

$$P_0 \sum_{r=0}^{N} a_r x^r + P_1 \sum_{r=0}^{N} r a_r x^{r-1} + P_2 \sum_{r=0}^{N} r(r-1) a_r x^{r-2} + \int_a^b k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) \tag{8}$$

Hence ,further simplification of equation (8), we obtained

$$\sum_{r=0}^{N} [P_0 a_r + P_1 (r+1) a_{r+1} + P_2 (r+1)(r+2) a_{r+2}] x^r + \int_a^b k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) \tag{9}$$

The integral part of equation (9) is evaluated and the left -over is then collocated at the point $x = x_k$, we obtained

$$\sum_{r=0}^{N} [P_0 a_r + P_1 (r+1) a_{r+1} + P_2 (r+1)(r+2) a_{r+2}] x_k^r + \int_a^b k(x_k,t) \sum_{r=0}^{N} a_r t^r dt = f(x_k) \tag{10}$$

Where,

$$x_k = a + \frac{(b-a)k}{N}, \; k = 1, 2, 3, \ldots, N-1 \tag{11}$$

Thus, equation (10) gives rise to (N-1) algebraic linear equation in (N+1) unknown constants. Two extra equations are obtained using equations (6) and (7) . Altogether, we have (N+1) algebraic linear equations in (N+1) unknown constants. These (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (4) to obtain the approximate solution.

IV. PERTURBED COLLOCATION METHOD BY POWER SERIES (PCMPS)

We used the method to solve equations (1)—(3) by substituting equation (4) into a slightly perturbed equation (1) to get

$$P_o y_N(x) + P_1 y'_N(x) + P_2 y''_N(x) + \int_a^b k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) \tag{12}$$

Where, τ_1 and τ_2 are two free tau parameters to be determined along with the constants $a_r (r \geq 0)$ and $T_N(x)$ is the Chebyshev polynomial of degree N in [a,b] defined by

$$T_{N+1}(x) = 2\left(\frac{2x-a-b}{b-a}\right) T_N(x) - T_{N-1}(x), \qquad N \geq 0 \tag{13}$$

Hence, further simplification of equation (12), we obtained

$$\sum_{r=0}^{N} [P_0 a_r + P_1 (r+1) a_{r+1} + P_2 (r+1)(r+2) a_{r+2}] x^r + \int_a^b k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) \tag{14}$$

The integral part of equation (14) is evaluated and the left-over is then collocated at the point $x = x_k$, we obtained

$$\sum_{r=0}^{N} [P_0 a_r + P_1 (r+1) a_{r+1} + P_2 (r+1)(r+2) a_{r+2}] x_k^r + \int_a^b k(x_k, t) \sum_{r=0}^{N} a_r t^r dt = f(x_k) + \tau_1 T_N(x_k) + \tau_2 T_{N-1}(x_k) \quad (15)$$

where,

$$x_k = a + \frac{(b-a)k}{N}, \ k = 1, 2, 3, \dots N-1 \quad (16)$$

Thus, equation (15) gives rise to (N+1) algebraic linear equations in (N+3) unknown constants. Two extra equations are obtained using equations (6) and (7). Altogether, we have (N+3) algebraic linear equations in (N+3) unknown constants. These (N+3) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants a_r (r ≥0) together with the parameters τ_1 and τ_2 which are then substituted back into equation (4) to obtain the approximate solution.

V. STANDARD COLLOCATION METHOD BY CANONICAL POLYNOMIALS (SCMCP)

We used the method to solve equations (1)-(3) by assuming canonical polynomial approximation of the form

$$y_N(x) = \sum_{r=0}^{N} a_r \Phi_r(x) \quad (17)$$

Where, x represents the independent variables in the problem, $a_r (r \geq 0)$ are the unknown constants to be determined and $\Phi_r(x)(r \geq 0)$ are canonical polynomials which should be constructed.

Thus, equation (17) is substituted into equations (1)-(3), we obtained

$$P_0 \sum_{r=0}^{N} a_r \Phi_r(x) + P_1 \sum_{r=0}^{N} a_r \Phi_r^{'}(x) + P_2 \sum_{r=0}^{N} a_r \Phi_r^{''}(x) + \int_a^b k(x,t) \sum_{r=0}^{N} a_r \Phi_r(t) dt = f(x) \quad (18)$$

Together with the conditions

$$\sum_{r=0}^{N} a_r \Phi_r(a) + \sum_{r=0}^{N} a_r \Phi_r^{'}(a) = A \quad (19)$$

and

$$\sum_{r=0}^{N} a_r \Phi_r(b) + \sum_{r=0}^{N} a_r \Phi_r^{'}(b) = B \quad (20)$$

Equation (18) is re-written as

$$P_0 a_0 \Phi_0(x) + P_0 a_1 \Phi_1(x) + \cdots + P_0 a_N \Phi_N(x) + P_1 a_0 \Phi_0^{'}(x) + P_1 a_1 \Phi_1^{'}(x) + \cdots + P_1 a_N \Phi_N^{'}(x) + P_2 a_0 \Phi_0^{''}(x)$$

$$+ P_2 a_1 \Phi_1^{''}(x) + \cdots + P_2 a_N \Phi_N^{''}(x) + \int_a^b k(x,t) \sum_{r=0}^{N} a_r \Phi_r(t) dt = f(x) \quad (21)$$

Hence, further simplification of equation (21), we obtained

$$[P_0 \Phi_0(x) + P_1 \Phi_0^{'}(x) + P_2 \Phi_0^{''}(x)] a_0 + [P_0 \Phi_1(x) + P_1 \Phi_1^{'}(x) + P_2 \Phi_1^{''}(x)] a_1 + \cdots + [P_0 \Phi_N(x) +$$

$$P_1 \Phi_N^{'}(x) + P_2 \Phi_N^{''}(x)] a_N + \int_a^b k(x,t) \sum_{r=0}^{N} a_r \Phi_r(t) dt = f(x) \quad (22)$$

The integral part of equation (22) is evaluated and the left-over is then collocated at the point x = x_k , we obtained

$$[P_0\Phi_0(x_k)+P_1\Phi_0^{'}(x_k)+P_2\Phi_0^{''}(x_k)]a_0+[P_0\Phi_1(x_k)+P_1\Phi_1^{'}(x_k)+P_2\Phi_1^{''}(x_k)]a_1+\cdots+[P_0\Phi_N(x_k)+P_1\Phi_N^{'}(x_k)$$

$$+P_2\Phi_N^{''}(x_k)]a_N+\int_a^b k(x_k,t)\sum_{r=0}^{N}a_r\Phi_r(t)dt=f(x_k) \qquad (23)$$

where

$$x_k=a+\frac{(b-a)k}{N},\ k=1,\ 2.,3.,\ \dots,\ N-1 \qquad (24)$$

Thus, equation (23) gives rise to (N-1) algebraic linear equations in (N+1) unknown constants. Two extra equations are obtained using equations (19) and (20). Altogether, we have (N+1) algebraic linear equations in (N+1) unknown constants. These (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (17) to obtain the approximate solution.

VI. PERTURBED COLLOCATION METHOD BY CANONICAL POLYNOMIALS (PCMCP)

We used the method to solve equations (1)-(3) by substituting equation (17) into a slightly perturbed equation (1) to get

$$P_o\,y_N(x)+P_1\,y'_N(x)+P_2\,y_N''(x)+\int_a^b k(x,t)\sum_{r=0}^{N}a_r\Phi_r(t)\,dt=f(x)+\tau_1T_N(x)+\tau_2T_{N-1}(x) \qquad (25)$$

Where, τ_1 and τ_2 are two free tau parameters to be determined along with the constants $a_r(r\ge 0)$ and $\Phi_r(x)$ is the canonical polynomial of degree N.

Hence, further simplification of equation (25), we obtained

$$[P_0\Phi_0(x)+P_1\Phi_0^{'}(x)+P_2\Phi_0^{''}(x)]a_0+[P_0\Phi_1(x)+P_1\Phi_1^{'}(x)+P_2\Phi_1^{''}(x)]a_1+\cdots+[P_0\Phi_N(x)+P_1\Phi_N^{'}(x)+P_2\Phi_N^{''}(x)]a_N$$

$$+\int_a^b k(x,t)\sum_{r=0}^{N}a_r\Phi_r(t)dt=f(x)+\tau_1T_N(x)+\tau_2T_{N-1}(x) \qquad (26)$$

The integral part of equation (26) is evaluated and the left-over is then collocated at the point $x = x_k$, we obtained

$$[P_0\Phi_0(x_k)+P_1\Phi_0^{'}(x_k)+P_2\Phi_0^{''}(x_k)]a_0+[P_0\Phi_1(x_k)+P_1\Phi_1^{'}(x_k)+P_2\Phi_1^{''}(x_k)]a_1+\cdots+[P_0\Phi_N(x_k)+P_1\Phi_N^{'}(x_k)$$

$$+P_2\Phi_N^{''}(x_k)]a_N+\int_a^b k(x_k,t)\sum_{r=0}^{N}a_r\Phi_r(t)dt=f(x_k)+\tau_1T_N(x_k)+\tau_2T_{N-1}(x_k) \qquad (27)$$

where

$$x_k=a+\frac{(b-a)k}{N+2},\ k=1,\ 2,\ 3\ ,\ \dots.\ N+1 \qquad (28)$$

Thus, equation (27) gives rise to (N+1) algebraic linear equations in (N+3) unknown constants. Two extra equations are obtained using equations (19) and (20). Altogether, we have (N+3) algebraic linear equations in (N+3) unknown constants. These (N+3) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants $a_r(r\ge 0)$ together with the parameters τ_1 and τ_2 which are then substituted back into equation (17) to obtain the approximate solution.

VII. CONSTRUCTION OF CANONICAL POLYNOMIALS

The canonical polynomials denoted by $\Phi_r(x)$ is generated recursively from equation (1) as follows:
Following [13], we define our operator as:

$$L\equiv P_2\frac{d^2}{dx^2}+P_1\frac{d}{dx}+P_0$$

Let $\quad L\Phi_r(x)=x^r$

Thus, $\quad Lx^r=P_2r(r-1)x^{r-2}+P_1rx^{r-1}+P_0x^r$

Implies, $\quad L\{L\Phi_r(x)\}=Lx^r\equiv P_2r(r-1)x^{r-2}+P_1rx^{r-1}+P_0x^r$

$$L\{L\Phi_r(x)\}=P_2r(r-1)L\Phi_{r-2}(x)+P_1rL\Phi_{r-1}(x)+P_0L\Phi_r(x)$$

We assumed that L^{-1} exists, then

$$x^r = P_2 r(r-1)L\Phi_{r-2}(x) + P_1 rL\Phi_{r-1}(x) + P_0 L\Phi_r(x)$$

Implies,

$$\Phi_r(x) = \frac{1}{P_0}\left\{x^r - P_2 r(r-1)\Phi_{r-2}(x) - P_1 r\Phi_{r-1}(x)\right\};\ r \geq 0,\ P_0 \neq 0 \qquad (29)$$

Hence, equation (29) is our constructed recursive canonical polynomials used in this work.
Remarks:

i. First order linear Integro-Differential Equation: For the purpose of our discussion, we set P_2=0 in equation (1) and this resulted to first order linear Integro-Differential equation considered in this work together with the initial condition y(a)=A (30)

ii. Errors: For the purpose of this work, we have defined maximum error used as

Maximum Error = $\max_{a \leq x \leq b} |y(x) - y_N(x)|$

8. Numerical Examples

Examples 1: Consider the first order linear integro-differential equation

$$y'(x) + 2y(x) + 5\int_0^x y(t)dt = 1 \qquad (31)$$

with initial condition
y(0)=0

The exact solution is given as $y(x) = \frac{1}{2}e^{-x}\sin(2x)$.

Table 1: Absolute maximum errors for example 1

N	Standard Collocation Method by Power Series (SCMPS)	Standard Collocation Method by Canonical Polynomials(SCMCP)	Perturbed Collocation Method by Power Series (PCMPS)	Perturbed Collocation Method by Canonical Polynomials (PCMCP)
4	3.30842E-4	8.01922E-2	2.80105E-5	9.84836E-4
6	1.77942E-5	3.48756E-4	5.48351E-6	1.91790E-6
8	7.34987E-6	5.78564E-6	2.78564E-7	9.23458E-8

Example 2:

Consider the first order linear integro differential equation

$$y'(x) = y(x) - Cos(2\pi x) - 2\pi\, Sin(2\pi x) - \frac{1}{2}Sin(4\pi x) + \int_0^1 Sin(4\pi x + 2\pi t)\, y(t)\, dt \qquad (32)$$

together with the initial condition

$$y(0) = 1$$

The exact solution is given as:

$$y(x) = Cos(2\pi x)$$

Table 2: Absolute maximum errors for example 2.

N	Standard collocation method by Power series(SCMPS)	Standard collocation method by canonical polynomials(SCMCP)	Perturbed collocation method by Power series(PCMPS)	Perturbed collocation method by canonical Polynomials(PCMCP)
4	7.48300E-2	1.86680E-3	8.83939E-3	9.37068E-4
6	1.52471E-2	3.16809E-4	6.39096E-3	2.13246E-5
8	8.76953E-3	1.67845E-5	3.67589E-4	1.03421E-6

Example 3: Consider the second-order linear integro-differential equation

$$y''(x) = 9y(x) + \frac{e^{-15} - 1}{3} + \int_0^5 y(t)dt \qquad (33)$$

together with the boundary conditions

$y(0) = 1 \quad and \quad y(1) = e^{-3}$

The exact solution is given as

$y(x) = e^{-3x}$

Table 3: Absolute maximum errors for example 3.

N	Standard Collocation Method by Power Series (SCMPS)	Standard Collocation Method by Canonical Polynomials(SCMCP)	Perturbed Collocation Method by Power Series (PCMPS)	Perturbed Collocation method by Canonical Polynomials(PCMCP)
4	4.86680E-2	2.02310E-2	1.86433E-3	2.13172E-4
6	1.16878E-2	5.16037E-3	2.17081E-4	2.05136E-5
8	8.45834E-3	1.67452E-4	7.45801E-5	1.89561E-7

VIII. DISCUSSION OF RESULTS AND CONCLUSION

Integro – Differential equations are usually difficulty to solve analytically. In many cases, it is required to obtain the approximate solutions. In this work, we proposed perturbed Collocation by Canonical polynomials for first and second orders linear In tegro Differential Equations and comparison were made with the Standard Collocation Method by Power Series and Canonical Polynomials as the the basis functions.

The comparison certifies that Perturbed Collocation Method gives good results as these are evident in the tables of results presented.

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