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# Modified Tj's Method for Yield Line Analysis and Design of Slabs

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***Abstract: -*** The Yield line method is widely used for the analysis and design of reinforced, concrete slabs. The method here described relies on three parameters - the Geometry of the plate, the length of Yield lines & the orientation of the Yield lines. Here the governing equation is broken into integrable parts such that the internal work is taken as the sum of all Yield lines including the negative Yield lines multiplied by (2 + number of nodal moments) and the average moment (ḿ). The external work is taken as the product of the load and the sum of the volume of each integral part with the deflection taken as the length of the positive Yield lines at the segment. The solution for the average bending moment over the Yield lines reduces the solution of plates to simple geometry. Results from the method compares to those of the work method and equilibrium method.

***Keywords: -*** *Equilibrium method, Nodal moments, Pyramidal factor and Prism factor, Yield lines, Work method.*

## I. INTRODUCTION

Studies in the yield- line theory of concrete slabs which have largely avoided the question of the distribution of the support reaction were addressed by Johnarry [1]. A great deal of researchers [2],[3],[4],[5]and [6] has worked extensively on Johansen discovery with the aim of tying the yield line theory with more classical plastic theory. It appears that none of the researchers has used or considered the length of the yield line in each segment as having all the characteristics required to determine the internal work and external work which in turn gives the load or carrying capacity of the slab. The work of Johnarry[1] gave an insight into further research which revealed that the Geometry of the plate and the length of the yield lines are just enough for the analysis and design of slabs.

## II. METHOD

A review of Johnarry's work [1] (referred in the proposed method as the TJ's method) has the yield-line as the s-axis and its normal n-axis. The yield –line method equation is

$$\int md\left(\frac{dw}{dn}\right) = \int qdsdndw.......... (1)$$

Where dw is the elemental deflection
dividing through by (dw/dn), we have

$$\int mds = \int qdsdn\int dn...............(2)$$

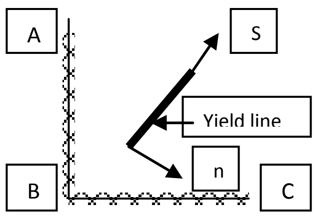



Fig 1-Analysis axes s & n

Equation (2) requires the slab to be divided into continuously integrable components, which must be the same rigid components produced by the yield – lines. The length of the yield lines in each component will be relied, upon to achieve this.

Integration is implied in equation (2) and this must be carried out along the yield lines for example, in fig 1 the yield – line BD relates to the support axes AB and BC. For areas $A_1$ and $A_2$ on both sides of the yield – line

$M_F = 2mL_{BD} = q\,A_1\,(a_R - a_q)_1 + qA_2\,(a_R - a_q)_2$ ……(3)

Where $M_F$ is the total amount of forces on both sides of a yield line, $a_R$ is the lever arm to the load centre and $a_R$ is the lever – arm to the reaction centre. Note that the yield – line length $L_{BD}$ is used twice in equation 3. For the case of a UDL called q, the reaction $R_j$ is

$R_j = qA_j$…………(4)

### 2.1 Theoretical formulation of proposed method

This study has shown that it is unnecessary to begin to look for $a_R$ and $a_q$ as proposed by the TJ's method rather the value of $(a_R - a_q)$, should always be taken as the length of the yield line at the segment divided by the pyramidal factor or prism factor. In case of a Udl. this means that eqn 3 is modified as

$$M_F = qA1\left(\frac{LBD}{F1}\right) + qA_2\left(\frac{LBD}{F_2}\right) + \ldots\ldots\ldots + qA_n\left(\frac{Lij}{Fn}\right) \ldots\ldots(5)$$

For pyramidal factor F = 3 , eqn 5 becomes

$$M_F = qA_1\frac{LBD}{3} + qA_2\frac{LBD}{3} \ldots\ldots\ldots\ldots\ldots(6)$$

For prism factor F = 2

$$M_F = qA_1\frac{LBD}{2} + qA_2\frac{LBD}{2} \ldots\ldots\ldots\ldots\ldots(7)$$

In the method here described the equations 6 & 7 are used in combination in most cases

#### 2.1.1 Pyramidal factor

The volume of a rectangular pyramid is given by

$$vol = \frac{1}{3}lbh = \frac{l}{3}area$$

The volume of a right angle triangular pyramid is given by

$$vol = \frac{1}{2}Lb\frac{h}{3} = \frac{lbh}{6}$$

#### 2.1.2 Prism factor

The volume of a prism is given by

$$vol = lb\frac{h}{2} = \frac{lbh}{2} = \frac{h}{2}area$$

In fig 2 below

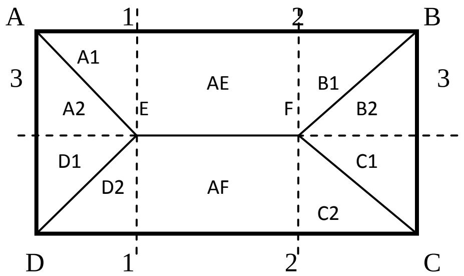



Fig. 2 Segments A1E 3, D1E3, B2F3 and C2F3 are pyramids, while segment 1221 is a prism.

$$M_F = q\left[\frac{LAE}{3}(A_{A1}+A_{A2}) + LDE\left(\frac{AD_1+AD_2}{3}\right) + LBF\left(\frac{AB_1+AB_2}{3}\right) + LCF\left(\frac{AC_1+AC2}{3}\right) + \frac{LEF}{2}(AE+AF)^3\right] \ldots\ldots\ldots\ldots\ldots(8)$$

The left hand side

$MF = 2ML_{BD}$ ………………………………..(9)

Needs modification as reported by researchers [1][2]. Here it has been established that when a yield –line meets a free edge, a nodal moment will occur at the dip. This fact is employed in this method to mean than equation 9 should be modified to carter for cases with free ends. Thus it becomes.

$M_F$ = M (2 + no of free edges) $L_{BD}$ ……(10)

Equation 10 is the required internal work in the slab, while the right hand side of equations (5), (6), (7) & (8) are the external work done in the slab.

### 2.2 Point Loads

In the case of point loads, the internal work remains the same, but the external work needs to be modified as follows equation 8 becomes.

$$M_F = \frac{P}{L}\left[LAE\left(A_{A_1} + A_{A_2}\right) + LBD\left(A_{D_1} + A_{D_2}\right) + \ldots\ldots\ldots.LEF\left(A_{A_E} + A_{A_E}\right)\ldots\ldots\ldots\ldots\ldots(11)\right]$$

### 2.3 Clamped Edges and Re-Entrant Corners

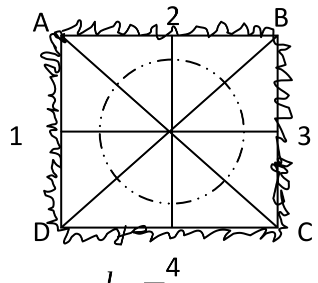
Here the internal work shall include the sum of positive and negative Yl, but the external work shall only deal with volume displaced by the positive Yl alone.

## III Results of Interest

It is assumed that, the user of the proposed method is familiar with the theory of yield lines and so should be able to draw yield lines for a given supported slab using the rules of Yl postulation[5][6].

### 3.1 Simply Supported Square Slab Loaded Uniformly

solution:



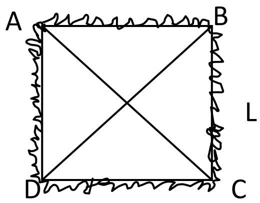



FIG 3

**Internal Work** $M_{f_m} = m\frac{l}{2}\sqrt{2}.4(2+0) = 4ml\sqrt{2}$

**External Work** $M_{F_E} = q\left[\frac{l}{2}.\frac{l}{2}.\frac{l}{2}\sqrt{2}.\frac{1}{3}X4\right] = \frac{qL^3\sqrt{2}}{6}$

Equating internal work to external work

$$m = \frac{ql^2}{24}$$

$M=0.0417ql^2$

$m_{classical}=0.0479ql^2$

### 3.2 Case 2: Clamped square slab loaded uniformly

$$M_{F_{internal}} = m\left[\frac{l\sqrt{2}}{2}x4 + 4l\right][2+0]$$

$$= 8ml\left(1 + 0.5\sqrt{2}\right)$$

$$M_{f_{external}} = q\left(\frac{l}{2}.\frac{l}{2}\sqrt{2}.\frac{1}{3}x4\right) = \frac{ql^3\sqrt{2}}{6}$$

Equating

$$m = \frac{ql^2}{48}\frac{\sqrt{2}}{\left(1+0.5\sqrt{2}\right)} = 0.0172589ql^2$$

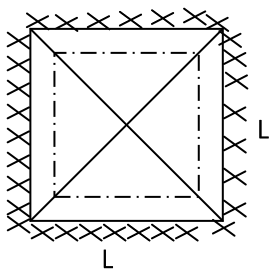


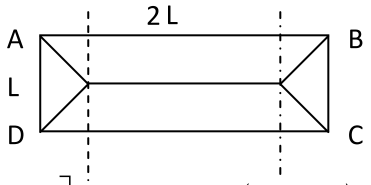
Fig 4

$$m = \frac{ql^2}{57.941}$$

**3.3 Case 3: Simply supported rectangular slab loaded uniformly**

Solution

Internal work




$$M_{f\,\mathrm{int}} = m\left[4\frac{l}{2}\sqrt{2}+l\right](2+0) = 2ml\left(1+2\sqrt{2}\right)$$

$$M_{ext} = q\left(\frac{1}{2}.\frac{l}{2}.\frac{l}{2}\sqrt{2}.\frac{l}{3}x4 + l.l.\frac{l}{2}\right) = \frac{ql^3}{6}\left(3+\sqrt{2}\right)$$

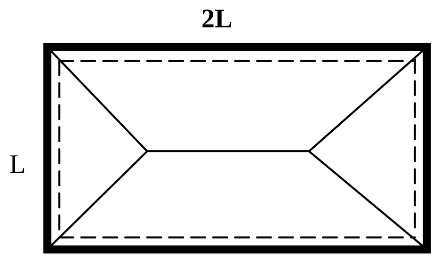
Equating internal work to external work

$$m = \frac{ql^2}{12}\frac{\left(3+\sqrt{2}\right)}{\left(1+2\sqrt{2}\right)} = 0.09608414ql^2$$

$$m = \frac{ql^2}{10.408}$$

**Mclassical = 0.10397ql$^2$**

**3.4 Case 4 All round clamped rectangular slab loaded uniformly**




Internal work $= M\left[4.\frac{1}{2}\sqrt{2}+L+L+L+2l+2l\right][2+0]$

$$= 4ML\left[\sqrt{2}+3.5\right]$$

External work same as that in case 4

External work $= \frac{ql^3}{6}\left(3+\sqrt{2}\right)$

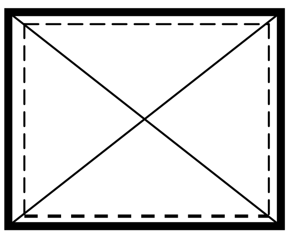
Equating

M = 0.0374273ql$^2$

$$m = \frac{ql^2}{26.9185}$$

$$M_{dassical} = 0.0365\,ql^2$$

**3.5 Case 5: Simply supported square slab loaded with a point**



$$M_{f\,\mathrm{internal}} = ml\frac{\sqrt{2}}{2}x4(2+0) = 4l\sqrt{2}m$$

$$M_{fexternal} = \frac{p}{l}\left(\frac{l}{2}.\frac{l}{2}\right)x4x\frac{l\sqrt{2}}{2}$$

Equating

$$m = \frac{pl2\sqrt{2}}{2x42\sqrt{2}} = \frac{pl}{8}$$

**3.6 Case 6: Simply supported rectangular slab loaded with a Point load**

$$M_{F\,internal} = M\left[\frac{l\sqrt{2}}{2}x4 + L\right][2+0]$$

$$= 2ml\left(2\sqrt{2}+1\right)$$

$$M_{F\,external} = \frac{P}{l}\left[\frac{l}{2}.\frac{l}{2}\right]4.\frac{l}{2}\sqrt{2} + \frac{p}{l}.lll$$

$$= Pl^2\left(1+0.5\sqrt{2}\right)$$

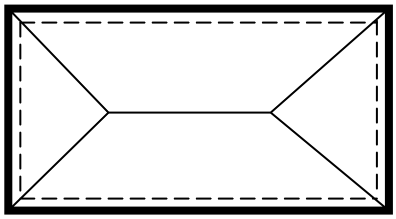
Equating

$$m = \frac{pl}{2}\frac{\left(1+0.5\sqrt{2}\right)}{\left(1+2\sqrt{2}\right)}$$

$$m = \frac{pl}{4.4852}$$

M = 0.222951pl

**3.7 Case 7: Clamped square plate loaded with a point load**



$$M_{F\,internal} = M\left[\frac{L\sqrt{2}}{2}x4 + 4L\right][2+0]$$

$$= 8ml\left(1+0.5\sqrt{2}\right)$$

$$Mext. = \frac{Pl^2\sqrt{2}}{2}(\,fromcase\,5)$$

Equating

$$m = \frac{Pl^2}{16l}\frac{\sqrt{2}}{\left(1+0.5\sqrt{2}\right)}$$

M = 0.051777pL

M = pl/19.31371

**3.8 Case 8: Clamped rectangle plate loaded with a point load.**

$$M_{finternal} \quad M\left[\frac{L\sqrt{2}}{2}X\,4 + L + l + l + 2l + l\right][2+0]$$

$$= 4ml\left(\sqrt{2}+3.5\right)\text{.........}\,(case4)$$

$$M_{fexternal} = Pl^2\left(1+0.5\sqrt{2}\right)\text{.....................}(case6)$$

Equating

$$m = \frac{pl}{4}\frac{\left(1+0.5\sqrt{2}\right)}{\left(3.5+\sqrt{2}\right)}$$

= 0.08685pl

$$m = \frac{pl}{11.5147}$$

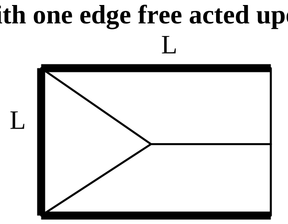
**3.9 Case 9: Square plate s- s on three sides with one edge free acted upon by a UDL q**

Solution

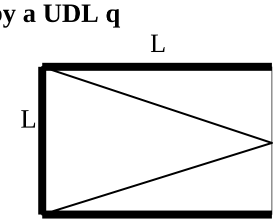
Internal work $= \left[\frac{2l}{2}\sqrt{2} + L\right][2+1]$

$$= 3ml\left[\sqrt{2}+1\right]$$

$$= 3ml\left(1+\sqrt{2}\right)$$




Case A




Case B

External work

$$= q\left[\frac{l}{2}.\frac{l}{2}.\frac{l\sqrt{2}}{2}.\frac{1}{3}.2 + l.\frac{l}{2}.\frac{l}{2}.\frac{1}{2}\right]$$

$$= \frac{ql^3}{24}\left[3+2\sqrt{2}\right]$$

Equating

$$m = \frac{ql^2}{72}\left[\frac{3+2\sqrt{2}}{1+\sqrt{2}}\right]$$

m = 0.0498025ql$^2$

$$m = \frac{ql^2}{20.0793}$$

**Case B**

$$m_{\text{int.}} = m.2.\sqrt{l^2} + \frac{l^2}{4}x[2+1]$$

$$= 6ml\sqrt{1.25}$$

$$m_{ext} = ql.l\,\frac{l\sqrt{1.25}}{3} = \frac{ql^3\sqrt{1.25}}{3}$$

Equating

$$m = \frac{ql^2}{18}$$

M = 0.05556ql$^2$

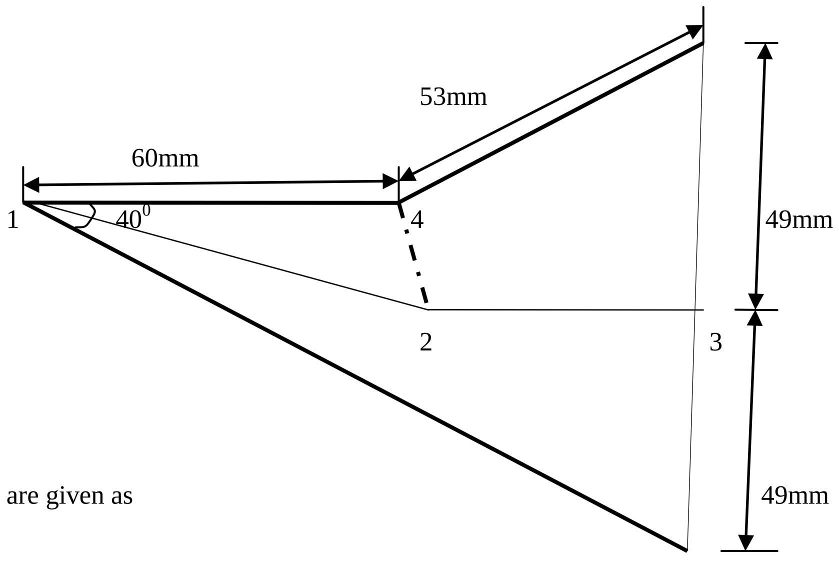
The results for case A and Case B for plate S-S on three sides with one edge free acted upon by a Point load Using the described method are respectively

$$m = \frac{pl}{12} \text{ and } m = \frac{pl}{6}$$

**3.10 Case 10 Irregular plates**

A typical example, which was earlier solved using the work method and the reaction method shown below, shall be solved using the described method

Solution




The lengths of the line measured are given as

YL 1-2 = 69mm
YL 2.3 = 37. 5mm
YL 2-4 = 24

Internal Energy $= (69+37.5+24)(2+1)m = 391.5m$

To obtain the external Energy, three cases where considered to demonstrate the method. The sharing of the area of influence each line such that the division line is perpendicular to yield line 2-3 for case 1; perpendicular to yield line 1-2 for case 2 and the simplest form was sharing the area equally to both yield line 2-3 and yield line 1-2. Remember that yield line 2-4 is a negative one and does not influence the external energy as described earlier.

**3.10.1 case 1**

External Energy $= \left(\frac{1}{2}.68.51.5.\frac{1}{2}.7.3.5\right)\frac{69}{2}q + \frac{(55+98)}{2}.37.5.\frac{37.5}{2}q$

$$= 114621.11875q$$

Equating

$$m = 292.77q$$

**3.10.2 case 2 perpendicular to YL 1-2 = 69m**

$$Mext = \left(\frac{1}{2}.69.51 + \frac{1}{2}.14.6\right)\frac{69}{2}q + \left(\frac{45+30}{2}\right).\left(\frac{98+55}{2}\right).\frac{37.5}{2}q$$

$$= 62151.75q + \left(\frac{45+30}{2}\right).\left(\frac{98+55}{2}\right).\frac{37.5q}{2}$$

= 115940.8125

$$m = \frac{115940.8125}{391.5} = 296.145115q$$

M = 296.15q

M=313.6q (work method)

% diff = - 5.57%

**3.10.3 case 3 equal half area**

$$mext = \left[\frac{1}{2}.69.51 + \frac{1}{2}.14.6 + \frac{75}{2}.\frac{153}{2}\right]\frac{1}{2}\left[\frac{69}{2} + \frac{37.5}{2}\right]q$$

= 124345.4063q

$M_{int}$ = 391.5m

Equating

$$m = \frac{124345.4063}{391.5}$$

M = 317. 6q

% diff = + 1.28%

## IV CONCLUSION

The reliable results obtained from the method herein described when compared to classical methods have given an insight to another view of yield line analysis. The main advantage of the proposed method is in the ease with which it can handle irregular plates. The example treated in case 10 is a clear case which could be handled with such ease and accuracy. Several plates in Engineering practice have shown serious yield lines which is beyond the realm of elastic analysis. The existing methods of handling this which is in fracture mechanics are not easily comprehended by those in practice who need simple but accurate hand method to handle such problems. The work can be extended to solving problems of cracks in structural walls and slabs, highway pavements, earth roads and bond walls around storages with high temperature gradients among others. This is possible because only the lengths of the yield lines and the geometry of the plate are required for their solutions. The work is limited to the average moment required to create the mechanism and by extension the maximum moment in each of the yield lines. The deflection of the plates are not discussed in this paper but can be researched into.

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