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Common Fixed Point Theorems for Sequence Of Mappings Under Contractive Conditions In Symmetric Spaces

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Abstract: - The main purpose of this paper is to obtain common fixed point theorems for sequence of mappings under contractive conditions which generalizes theorem of Aamri [1].

Keywords And Phrases: - *Fixed point, Coincidence point, Compatible maps, weakly compatible maps, NonCompatible maps Property (E.A).*

I. INTRODUCTION

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. Hicks [2] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set X is a nonnegative real valued function d on X × X such that (i) d(x, y) = 0 if, and only if, x = y, and (ii) d(x, y) = d(y, x). Let d be a symmetric on a set X and for r > 0 and any x ∈ X, let B(x, r) ={y ∈ X: d(x, y) < r}. A topology t (d) on X is given by U ∈ t (d) if, and only if, for each x ∈ U, B(x, r) ⊂ U for some r > 0. A symmetric d is a semi-metric if for each x ∈ X and each r > 0, B(x, r) is a neighbourhood of x in the topology t(d). Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology t (d).

II. PRELIMINARIES

Before proving our results, we need the following definitions and known results in this sequel.

Definition 2.1([3]) let (X, d) be a symmetric space. (W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply x = y. (W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.
Definition 2.2([4]) Two self mappings A and B of a metric space (X, d) are said to be weakly commuting if $d(AB_x, BA_x) \le d(A_x, B_x)$, ∀x ∈ X.
Definition 2.3([5]) Let A and B be two self mappings of a metric space (X, d). A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t ∈ X.
Remark 2.4. Two weakly commuting mappings are compatibles but the converse is not true as is shown in [5].
Definition 2.5 ([5]) Two self mapping T and S of a metric space X are said to be weakly compatible if they commute at there coincidence points, i.e., if $T_u = S_u$ for some u ∈ X, then $TS_u = ST_u$.
Note 2.6. Two compatible maps are weakly compatible. M. Aamri [6] introduced the concept property (E.A) in the following way.
Definition 2.7 ([6]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some t ∈ X.
Definition 2.8 ([6]). Two self mappings S and T of a metric space (X, d) will be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that if $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non-existent.
Remark 2.9. Two noncompatible self mappings of a metric space (X, d) satisfy the property (E.A).
In the sequel, we need a function $\varphi: IR^+ \to IR^+$ satisfying the condition $0 < \varphi(t) < t$ for each t > 0.

Definition 2.10. Let A and B be two self mappings of a symmetric space (X, d).A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t)$ $=\lim_{n\to\infty} d(Bx_n, t) = 0$ for some t ∈ X.
Definition 2.11. Two self mappings A and B of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.
Definition 2.12. Let A and B be two self mappings of a symmetric space (X, d). We say that A and B satisfy the property (E.A) if there exists a sequence (x_n) such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$ for some t ∈ X.
Remark 2.13. It is clear from the above Definition 2.10, that two self mappings S and T of a symmetric space (X, d) will be noncompatible if there exists at least one sequence (x_n) in X such that $\lim_{n\to\infty} d(Sx_n, t)$ $=\lim_{n\to\infty} d(Tx_n, t) = 0$ for some t ∈ X. but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or does not exist. Therefore, two noncompatible self mappings of a symmetric space (X, d) satisfy the property (E.A).
Definition 2.14. Let (X, d) be a symmetric space. We say that (X, d) satisfies the property (H_E) if given {x_n}, {y_n} and x in X, and $\lim_{n\to\infty} d(x_n, x) = 0$ *and* $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(y_n, x_n) = 0$
Note that (X,d) is not a metric space.
Aamri [1] prove the following theorems.
Theorem 2.15 (Aamri [1]). Let d be a symmetric for X that satisfies (W.3) and (H_E). Let A and B be two weakly compatible self mappings of (X, d) such that (1) $d(A_x,A_y) \le \varphi(\max\{d(B_x,B_y), d(B_x,A_y), d(A_y,B_y)\})$ for all $(x, y) \in X^2$, (2) A and B satisfy the property (E.A), and (3) AX ⊂ BX. If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point.
Theorem 2.16 (Aamri [1]). Let d be a symmetric for X that satisfies (W.3), (W.4) and (H_E). Let A, B, T and S be self mappings of (X, d) such that (1) $d(A_x,B_y) \le \varphi(\max\{d(S_x, T_y), d(S_x,B_y), d(T_y,B_y)\})$ for all $(x, y) \in X^2$,
(2) (A, T) and (B,S) are weakly compatibles, (3) (A, S) or (B, T) satisfies the property (E.A), and
(4) AX ⊂ TX and BX ⊂ SX. If the range of the one of the mappings A, B, T or S is a complete subspace of X, then A, B, T and S have a unique common fixed point.

III. MAIN RESULTS

In this section we prove common fixed point theorem for sequence of mappings that generalizes Theorem 2.16.
Theorem 3.1. Let d be a symmetric for X that satisfies (W.3) (W.4) and (H_E). Let {A_i}, {A_j}, S and T be self maps of a metric space (X, d) such that
(1) $d(A_ix,A_jy) < \max\{d(S_xT_y), d(A_ix,S_x), d(A_jy,T_y), d(A_ix,T_y), d(A_jy, S_x)\}$ for all $(x,y)\in X^2$,(i≠j),
(2) (A_i,S) or (A_k,T) are weakly compatibles. (3) (A_i, S) or (A_jT) ,(i≠j) satisfies the property(E.A) and
(4) $A_iX \subset TX$ and $A_jX \subset SX$ for (i≠j)
If the range of the one of the mappings {A_i}, {A_j}, S or T is a complete subspace of X,
then (I) A_i and S have a common fixed point, ∀ i (II) A_j,(i≠j) and T have a common fixed point provided that (A_k, T) for some k > 1 is weakly compatible. (III) A_i,A_j, S (i≠j) and T have a unique common fixed point provided that (I) and (II) are true.
Proof. Suppose that (A_j,T) (i≠j) satisfies the property (E.A.).
=>There exists a sequence {x_n} in X such that $\lim_{n\to\infty} d(A_jx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ for (i≠j) and for some t∈X. Since $A_jX \subset SX$ (i≠j), there exists a sequence {y_n} in X such that $A_jx_n = Sy_n$.
Hence, $\lim_{n\to\infty} d(Sy_n, t) = 0$ (since, $\lim_{n\to\infty} d(A_jx_n, t) = 0$)
Let us prove that $\lim_{n\to\infty} d(A_iy_n, t) = 0$
It is enough to prove that $A_iy_n = A_jx_n$, (i≠j) and for sufficiently large n.
Suppose not, then using (1)
$d(A_iy_n,A_jx_n) < \max\{d(Sy_n,Tx_n), d(A_iy_n ,Sy_n), d(A_jx_n,Tx_n), d(A_iy_n ,Tx_n), d(A_jx_n ,Sy_n)\}$ for all $(x,y) \in X^2$, (i≠j)
$d(A_iy_n,A_jx_n) < \max\{d(A_jx_n,Tx_n), d(A_iy_n,A_jx_n), d(A_jx_n ,Tx_n), d(A_iy_n ,Tx_n)\}$ for all $(x,y) \in X^2$,(i≠j),
For sufficiently large n, { since, $A_jx_n=Sy_n$}
$d(A_iy_n,A_jx_n) < \max\{ d(A_iy_n,A_jx_n), d(A_iy_n ,A_jx_n)\}< d(A_iy_n ,A_jx_n)$ {Since, $A_jx_n= Tx_n$ as n→∞}(By H_E)
=><= $A_iy_n \neq A_jx_n$ for (i≠j)
$\lim_{n\to\infty} d(A_iy_n,A_jx_n)=0$ By(W.2),we deduce that $\lim_{n\to\infty} d(A_iy_n, t)=0$.
Suppose SX is a complete subspace of X.Then t = Su for some u∈X.
Therefore, $\lim_{n\to\infty} d(A_iy_n ,S_u)= \lim_{n\to\infty} d(A_jx_n ,S_u)= \lim_{n\to\infty} d(Tx_n,S_u)$
$= \lim_{n\to\infty} d(Sy_n,S_u)=0$ (i≠j)
Using (1),it follows $d(A_iu,A_jx_n) < \max\{d(S_u,Tx_n), d(A_iu,S_u), d(A_jx_n ,Tx_n), d(A_iu,Tx_n), d(A_jx_n ,S_u)\}$ for sufficiently large n, (i≠j)
$d(A_iu,S_u) < \max\{d(A_iu,S_u), d(A_iu,S_u)\}$(i≠j),
$<d(A_iu,S_u)$ ∀ i =><= when $A_iu \neq S_u$ ∀ i
Therefore, $A_iu=S_u$ ∀ i
This means that A_i and S have coincidence point. But (A_i, S) ∀ i is weakly compatible.

$SA_iu = A_iS_u$ $\forall$ i and then $A_iA_iu = A_iS_u = SA_iu = SS_u$. $\forall$ i
Suppose $A_iX \subset TX$ $\forall$ i
=>There exists $v \in X$ such that $A_iu=T_v$ $\forall$ i
=> $A_iu=S_u=T_v$ $\forall$ i
To prove that $T_v= A_jv$, $(i\neq j)$
Suppose $T_v \neq A_jv$,then
(1)=> $d(A_iu,A_jv) < \max\{d(S_u,T_v), d(A_iu, S_u), d(A_jv,T_v), d(A_iu,T_v), d(A_jv,S_u)\}$
$= \max\{d(T_v,T_v), d(S_u, S_u), d(A_jv,T_v), d(T_v,T_v), d(A_jv,T_v)\}$ $(i\neq j)$
$= \max\{d(A_jv,T_v), d(A_jv,T_v)\}$ $(i\neq j)$
$= d(A_jv,T_v) = d(A_jv,A_iu)$, $(i\neq j)$
Therefore $(A_iu,A_jv)<d(A_jv,A_iu)$ $(i\neq j)$
=><= Therefore $A_iu=A_jv$ $(i\neq j)$
=>$A_jv=A_iu=T_v$ Therefore,$A_jv=T_v$ for $i\neq j$
=>$A_iu=S_u=T_v=A_jv$, $i\neq j$
But (A_k,T) is weakly compatible for some k>1
$A_kT_v=TA_kv$ for some k>1 and $TT_v=TA_kv=A_kT_v=A_kA_kv$, for some k>1
We shall prove that A_iu is a common fixed point of A_i and S $\forall$ i
Suppose $A_iu\neq A_iA_iu$ $\forall$ i
$d(A_iu,A_iA_iu) =d(A_jv,A_iA_iu)$ (since, $A_jv=A_iu$) $(i\neq j)$
$d(A_iA_iu,A_jv) < \max\{d(SA_iu,T_v), d(A_iA_iu, SA_iu), d(A_jv,T_v), d(A_iA_iu,T_v), d(A_jv,SA_iu)\}$ $(i\neq j)$
$= \max\{d(A_iA_iu,A_jv), 0,0, d(A_iA_iu,A_jv), d(A_jv,A_iA_iu)\}$ $(i\neq j)$
$= d(A_iA_iu,A_jv)$ Therefore, $d(A_jv,A_iA_iu)< d(A_iA_iu\ A_jv)$
=><=
Therefore, $A_iA_iu =A_jv$ $(i\neq j)$
=> $A_iA_iu =A_iu=SA_iu$ (since, $A_iA_iu =SA_iu$)
=> A_iu is a common fixed point of A_i and S. $\forall$ i This proves (I).
To prove that $A_kv = A_iu$ for some k>1 is a common fixed point of A_j $(i\neq j)$ and T
Suppose $A_kv\neq A_jA_kv$, then
$d(A_kv,A_jA_kv)= d(A_iu,A_jA_kv)$
$< \max\{d(S_u,TA_kv), d(A_iu,S_u), d(A_jA_kv ,TA_kv), d(A_iu,TA_kv), d(A_jA_kv,S_u)\}$
$= \max\{d(A_iu,A_jA_kv),0, d(A_jA_kv,A_jA_kv), d(A_iu, A_jA_kv), d(A_jA_kv\ A_iu)\}$ (since,$A_jv=T_v$)
$= \max\{d(A_iu,A_jA_kv),0,0, d(A_iu, A_jA_kv), d(A_jA_kv ,A_iu)\}$
Therefore, $d(A_kv,A_jA_kv)< d(A_iu,A_jA_kv)$.
=><= (since,$A_iu=A_kv$)
Therefore, $A_iu=A_jA_kv$ ie., $A_kv=A_jA_kv=TA_kv$ (since,$A_jv=T_v$)
=>A_kv is the common fixed point of A_j and T. This proves (II)
Now,A_iu is a common fixed point of A_i and S. $\forall$ i
$A_kv=A_iu$ is the common fixed point of $A_j$and T for $i\neq j$
Therefore, A_iu is the common fixed point of A_j,Tand S for all j $(i\neq j)$
The proof is similar when TX is assumed to be complete subspace of X.
The cases in which A_iX or A_jX $(i \neq j)$ is a complete subspace of X are similar to
the cases in which SX or TX respectively is a complete space because $A_iX \subset TX$ and $A_jX\subset SX$ $(i\neq j)$.
Uniqueness. Suppose u, v are two fixed points of A_i, A_j $(i\neq j)$, TandS.
Then $A_iu = S_u = A_ju = T_u = u$, $(i\neq j)$ and $A_iv = A_jv = T_v = S_v = v$, $(i\neq j)$.Then
$d(u,v) = d(A_iu,A_jv)$ $(i\neq j)$
$< \max\{d(S_u,T_v),d(A_iu,S_u), d(A_jv,T_v), d(A_iu,T_v), d(A_jv, S_u)\}$
$= \max\{d(u,v),0,0,d(u,v), d(u,v)\}$
$=d(u,v)$.
Therefore, $d(u,v)=d(u,v)$
==><== when $u\neq v$.
Therefore, u=v.
ie., A_i, A_j,T and S have unique common fixed point for all i and j.
The following result due to Aamri [1] is a special case of the previous theorem 3.1.
Corollary 3.1. Let d be a symmetric for X that satisfies (W.3) (W.4) and (H_E).Let A_1, A_2, S and T be self mappings of a metric space (X,d) such that
(i) $d(A_1x,A_2y) < \max\{d(S_x,T_y), d(A_1x, S_x),d(A_1x,T_y),d(A_2,T_y), d(A_2y,S_x)\}$ for all $(x,y)\ \varepsilon\ X^2$,
(ii) (A_1, S) and (A_2, T) are weakly compatibles.
(iii)(A_1, S) or (A_2, T) satisfies the property (E.A.) and

(iv) $A_1X \subset TX$ and $A_2X \subset SX$. If the range of one of the mappings A_1, A_2, S or T is a complete subspace of X, then A_1, A_2, S and T have a unique common fixed point.

Proof. The proof of Corollary 3.1 follows from Theorem 3.1 by putting i = 1 and j=2.

Corollary 3.2. Let d be a symmetric for X that satisfies (W. 3), (W.4) of Wilson and (H_E).
Let A, B and T be self mappings of a metric space (X,d) such that
(i)AX, BX ⊂ TX.
(ii) (A, T) is weakly compatible,
(iii) (A,T) or (B,T) satisfies the property (E.A.),
(iv) $d(A_x,B_y) < \max\{d(T_x,T_y),d(A_x,T_x),d(B_y,T_y), d(A_x,T_y),d(B_y,T_x)\}$
If the range of one of the mappings A, B or T is a complete subspace of X, then
(I) A and T have a common fixed point,
(II) B and T have a common fixed point provided that (B, T) is weakly compatible.
(III) A, B, S and T have a unique common fixed point provided that (I) and (II) are true.

Corollary 3.3. .Let d be a symmetric for X that satisfies (W.3),(W.4)and(H_E).Let G, T be self mappings of a metric space (X,d) such that (i) $d(T_x,T_y) \le \phi(\max\{d(G_x,G_y), d(G_x, T_y),d(G_y,T_y),1/2[d(G_x,T_y)+ d(G_y,T_y)\}$ for all (x,y) ε X^2,
(ii)G and T are weakly compatibles, (iii)T and G satisfy the property (E.A), and
(iv)TX ⊂GX .If the range of one of the mappings G or T is a complete subspace of X,
then G and T have a unique common fixed point.

Corollary 3.4. Let d be a symmetric for X that satisfies (W.1) of Wilson and (H_E).
Let S and T be two weakly compatible self mappings of a metric space (X,d) such that
(i) $d(T_x,T_y) \le \phi(\max\{d(S_x,S_y),d(S_x,T_y),d(S_y,T_y),1/2[d(S_x,T_y)+d(S_y,T_y)]\}$ for all (x,y) ε X^2,
(ii) Sand T satisfy the property (E.A.) and
(iii) SX ⊂ TX. If the range of S or T is a complete subspace of X, then S and T have a unique common fixed point.

Theorem 3.2. Let d be a symmetric for X that satisfies (W.3),(W.4) and (H_E). Let A, B, T and S be self mappings of a metric space (X,d) such that (i) $d(A_x,B_y) < \alpha\, d(B_y,T_y)\{[1 + d(A_x,S_x)]/1 + d(S_x,T_y)\} + \beta[d(B_y,T_y)+ d(A_x,S_x)]+ \gamma[d(B_y,S_x)+ d(A_x,T_y)]+\delta\, d(S_x,T_y)$ for all (x,y) $x \in X^2$ with α, β, γ, δ≥0 and $\alpha + \beta + 2\gamma + \delta < 1$ (ii) (A,S) and (B,T) are weakly compatibles. (iii) (A, S) or (B, T) satisfies the property (E.A.) (iv) AX⊂TX and BX⊂SX. If the range of one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof. Suppose (B, T) satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Bx_n,t) = \lim_{n\to\infty} d(Tx_n,t) = 0$ for some t ∈ X. Since BX ⊂SX, there exists in X a sequence (y_n) in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} d(Sy_n,t) = 0$.
Let us show that $\lim_{n\to\infty} d(Ay_n,t) = 0$
It is enough to prove that $Ay_n = Bx_n$. Suppose not, by (1), we get
$d(Ay_n,Bx_n) < \alpha\, d(Bx_n,Tx_n)\{[1 + d(Ay_n,Sy_n)]/1 + d(Sy_n,Tx_n)\}+\beta[d(Bx_nTx_n)+ d(Ay_n,Sy_n)]+\gamma[d(Bx_n,Sy_n)+ d(Ay_n,Tx_n)]+\delta\, d(Sy_n,Tx_n)$,
$<\alpha d(Bx_n,Tx_n)\{[1 + d(Ay_n,Bx_n)]/1 + d(Bx_n,Tx_n)\}+\beta[d(Bx_n,Tx_n)+d(Ay_n,Bx_n)]+\gamma[d(Bx_n,Sy_n)+d(Ay_n,Tx_n)] + \delta\, d(Bx_n,Tx_n)$
For sufficiently large n,
$d(Ay_n,Bx_n)<0+ \beta[0+d(Ay_n,Bx_n)]+ \gamma[0+d(Ay_n,Tx_n) < \beta d(Ay_n,Bx_n)+ \gamma d(Ay_n,Tx_n)$
$= (\beta+ \gamma)\, d(Ay_n,Bx_n)$ (since, $\lim_{n\to\infty} d(Bx_n,t) = \lim_{n\to\infty} d(Tx_n,t) = 0$)
This is a contradiction, $\lim_{n\to\infty} d(Ay_n,Bx_n) = 0$
By (W.3), we deduce that $\lim_{n\to\infty} d(Ay_n,t) = 0$
Suppose that SX is a complete subspace of X. Then t = Su for some u∈ X
Subsequently, we have $\lim_{n\to\infty} d(Ay_n,S_u) = \lim_{n\to\infty} d(Bx_n,S_u)=\lim_{n\to\infty} d(Tx_n,S_u)=\lim_{n\to\infty} d(Sy_n,S_u) = 0$
Using (1),
$d(A_u,Bx_n) < \alpha\, d(Bx_n,Tx_n)\{[1 + d(A_u,S_u)]/1 + d(S_u,Tx_n)\}+\beta[d(Bx_n,Tx_n)+(d(A_u,S_u)]+\gamma[d(Bx_n,S_u)+ d(A_u,Tx_n)]+\delta\, d(S_u,Tx_n)$
Letting $n \to \infty$, we have$\lim_{n\to\infty} d(A_u,Bx_n) < \beta d(A_u,S_u) +\gamma d(A_u,S_u)$
$d(A_u,S_u)< (\beta+ \gamma)d(A_u,S_u)$.
This is a contradiction for $A_u \ne S_u$.
The weakly compatibility of A and S implies that
$AS_u = SA_u$and then $AA_u = AS_u = SA_u = SS_u$.
Since AX ⊂TX, there exists v∈ X such that $A_u = T_v$. Therefore $A_u = S_u = T_v$.
We claim that $T_v = B_v$. If not condition (1) gives

$d(A_u,B_v) < \alpha\, d(B_v,T_v) \{[1 + d(A_u, S_u)]/1 + d(S_u, T_v)\} + \beta[d(B_v,T_v) + d(A_u,S_u)]+\gamma[d(B_v,S_u)+ d(A_u,T_v)]+ \delta\, d(S_u,T_v) < \alpha\, d(B_v,A_u) \{[1 + 0]/(1+0)\}+\beta[d(B_v,T_v)+0]+\gamma[d(B_v,A_u)+ 0]+\delta\,(0)$

$d(A_u,B_v) < \alpha\, d(B_v,A_u)+\beta d(B_v,A_u)+\gamma d(B_v,A_u)$.

$d(A_u,B_v) < (\alpha+\beta+\gamma)d(B_v,A_u)$.

This is a contradiction for $A_u \neq B_v$.

Therefore $A_u = B_v$ and then $B_v = A_u = T_v$.

This implies that $A_u = S_u = T_v = B_v$.

But (B, T) is weakly compatible implies $BT_v = TB_v$ and $TT_v = TB_v = BT_v = BB_v$.

We shall prove that A_u is a common fixed point of A and S.

Suppose that $AA_u \neq A_u$.

$d(A_u,AA_u) = d(AA_u,B_v)$

$< \alpha\, d(B_v,T_v) \{ [1 + d(AA_u, SA_u)]/1 + d(SA_u, T_v)\}+\beta[d(B_v,T_v)+ d(AA_u,SA_u)]+\gamma[d(B_v,SA_u)+ d(AA_u,T_v)]+ \delta\, d(SA_u,T_v)$

$=\gamma[d(B_v,AA_u)+ d(AA_u,B_v)]+\delta\, d(AA_u,B_v)$

$= (2\gamma+ \delta)d(AA_u,B_v)$

This is a contradiction for $AA_u \neq B_v$.

Therefore $AA_u = B_v$ and then $AA_u = A_u = SA_u$ (since $AA_u = SA_u$)

Therefore A_u is a common fixed point of A and S.

To prove that $B_v = A_u$ is a common fixed point of B and T.

Suppose $B_v \neq BB_v$.

$d(B_v,BB_v) = d(A_u,BB_v)$

$< \alpha\, d(BB_v,TB_v) \{ [1 + d(A_u, S_u)]/1 + d(S_u, TB_v)\} +\beta[d(BB_v,TB_v) + d(A_u,S_u)]+\gamma[d(BB_v,S_u)+ d(A_u,TB_v)]+ \delta\, d(S_u,TB_v)$

$= \gamma[d(BB_v,A_u)+ d(A_u,BB_v)]+\delta\, d(A_u,BB_v)$

$= (2\gamma+ \delta)d(A_u,BB_v)= (2\gamma+ \delta)d(B_v,BB_v)$

which is a contradiction for $B_v \neq BB_v$.

Therefore $B_v = A_u = BB_v = TB_v$.

This means that B_v is a common fixed point of B and T.

Therefore, A_u is the common fixed point of A and S.

$B_v = A_u$ is the common fixed point of B and T.

Therefore, A_u is the common fixed point of A, B, T and S.

The proof is similar when TX is assumed to be a complete subspace of X.

The cases in which AX or BX is a complete subspace of X are similar to the cases in which SX or TX respectively is a complete space because $AX \subset TX$ and $BX \subset SX$.

Uniqueness. Suppose u, v are two fixed points of A, B, T and S.

Then $A_u = S_u = B_u = T_u = u$.

and $A_v = B_v = T_v = S_v = v$. Then for $u \neq v$, and then (1) gives $d(u,v)=d(A_u,B_v)$

$< \alpha\, d(B_v,T_v) \{ [1 + d(A_u, S_u)]/1 + d(S_u, T_v)\}+\beta[d(B_v,T_v)+(d(A_u,S_u)]+\gamma[d(B_v,S_u)+ d(A_u,T_v)]+\delta\, d(S_u,T_v) = \gamma[d(B_v,A_u)+ d(A_u,B_v)]+\delta\, d(A_u,B_v)$

$=(2\gamma+ \delta)d(A_u,B_v)= (2\gamma+ \delta)d(u,v)$.

This is a contradiction for $u \neq v$.Therefore u = v.

This means that A, B, T and S have unique common fixed point.

For three maps, we have the following result by altering the condition (i) in theorem 3.2.

Corollary 3.3. Let d be a symmetric for X that satisfies (W.3), (W.4) of Wilson and (H_E). Let A, B and S be self mappings of a metric space (X,d) such that

(i) $AX, BX \subset SX$,

(ii) (A, S) is weakly compatible.,

(iii)(A, S) or (B, S) satisfies the property (E.A.),

(iv) $d(A_x,B_y) < \alpha\, d(B_y,S_y) \{[1 + d(A_x, S_x)]/1 + d(S_x, S_y)\}+\beta[d(B_y,S_y) + d(A_x,S_x)]+\gamma[d(B_y,S_x)+ d(A_x,S_y)]+\delta\, d(S_x,S_y)$ for all (x,y) $x \in X^2$ with $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta < 1$. If the range of one of the mappings A, B or S is a complete subspace of X, then A, B and S have a unique common fixed point.

For two maps, we have the following result by altering the condition (i) in theorem of Aamri [1].

Theorem 3.3. . Let d be a symmetric for X that satisfies (W.3) of Wilson and (H_E).

Let S and T be weakly compatible self mappings of a metric space (X,d) such that

(i) $d(T_x,T_y) < \alpha\{ d(T_x, S_x)/1 + d(S_x, T_y)\} +\beta d(T_x,S_x)+\gamma[d(T_y,S_x)+ d(T_x,T_y)]+\delta\, d(S_x,T_y)$ for all (x,y) $x \in X^2$ with $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + 2\gamma + \delta < 1$. (ii)T and S satisfy the property (E.A.), (iii) $TX \subset SX$, If SX or TX is a complete subspace of X, then T and S have a unique common fixed-point.

Proof. Since T and S satisfy the property (E.A). Then there exists a sequence (x_n) in X such that $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ for some $t \in X$.
Therefore, by (H_E), we have $\lim_{n\to\infty} d(Tx_n, Sx_n) = 0$
Suppose that SX is a complete subspace of X.
Then $t=S_u$ for some $u \in X$.
We claim that $T_u=S_u$
By (1) we have $d(Tx_n,T_u) < \alpha\{d(Tx_n, Sx_n)/1 + d(Sx_n, T_u)\} + \beta(d(Tx_n,Sx_n)]+\gamma[d(T_u,Sx_n)+ d(Tx_n,T_u)]+ \delta d(Sx_n,T_u)$. Letting n->∞, we have
$\lim_{n\to\infty} d(Tx_n, T_u) < \lim_{n\to\infty}\{\gamma[d(T_u, Sx_n) + d(Tx_n,T_u)]+\delta d(Sx_n,T_u)\}$
$d(S_u,T_u)<2 \gamma d(S_u,T_u)+ \delta d(S_u,T_u)=(2\gamma +\delta) d(S_u,T_u)$
This is a contradiction $S_u \neq T_u$. Therefore, $S_u = T_u$.
Since S and T are weakly compatible, $ST_u = TS_u$ and therefore $TT_u = TS_u = ST_u = SS_u$.
Let us prove that T_u is a common fixed point of T and S. Suppose $T_u \neq TT_u$,
Then $d(T_u,TT_u) < \alpha\{ d(T_u, S_u)/1 + d(S_u, TT_u)\} + \beta(d(T_u,S_u)]+\gamma[d(TT_u,S_u)+ d(T_u,TT_u)]+\delta d(S_u,TT_u)$
$<(2\gamma +\delta)d(T_u,TT_u)$
This is a contradiction for $T_u \neq TT_u$.
Therefore, $T_u=TT_u$ and $ST_u=TT_u=T_u$.
The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.
Uniqueness. Suppose T_u, T_v are two fixed points of T and S with $T_u \neq T_v$. Then
$d(T_u,T_v) < \alpha\{d(T_u, S_u)/1 + d(S_u, T_v)\} + \beta(d(T_u,S_u)]+\gamma[d(T_v,S_u)+ d(T_u,T_v)]+\delta d(T_u,T_v)$
Therefore $d(T_u,T_v)<(2\gamma +\delta)d(T_u,T_v)$.
This is a contradiction for $T_u \neq T_v$
Therefore, $T_u=T_v$ and hence, T and S have unique common fixed point.

REFERENCES

[1] M. Aamri and D. El. Moutawakil, common fixed point theorems under contractive conditions in symmetric spaces Appl.Math, 3 (2003) 156-162.
[2] T. L. Hicks, Fixed point theory in symmetric spaces with applications to probabilistic spaces, Nonlinear Analysis, 36(1999), 331—344.
[3] W. A. Wilson, on semi-metric spaces, Amer. J. Math., 53(1931), 361—373.
[4] S. Sessa, on a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd), 32(46) (1982), 149—153.
[5] G. Jungck, Compatible mappings and common fixed points, Intl. J. Math. Sci., 9 (1986), 771-779.
[6] M. Aamri and D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Theory and App., 270 (2002) 181-188.