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Initial Value Problem for Nonlocal Impulsive Integro-Differential Equations Involving Fractional Derivative of order θ

S. Suresha*, G. Thamizhendhib

aDepartment of Mathematics, Kongu Arts and Science College, Erode
. bDepartment of Mathematics, Sri Vasavi College, Erode.

ABSTRACT *: In this paper, we study the existence of solutions of the initial value problem for impulsive fractional integro differential equations involving nonlocal conditions. By using the fixed point theorems, the existence results are proved.*
KEYWORDS: *Existence, Fractional differential equations, Fractional Derivative and integral, fixed point theorem. AMS Subject Classification: 34A12, 34A08, 26A33, 47H10.*

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I. INTRODUCTION

One important and interesting area of research of fractional differential equations is a new branch of mathematics by valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, elec- trochemistry, control, porous media, electromagnetic, etc. (see [14, 15]) and reference therein. Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant de- velopment in impulsive theory especially in the area of impulsive differential equations with fixed moments and the references therein [1, 7–13, 18, 19, 21]. In [2, 4, 6] M. S. Abdo et. al., studied the fractional integro-differential equation with Caputo fractional derivative and Ψ- Hilfer fractional derivative, continuous dependence for fractional neutral functional differential equations.

*sureshkongucas@gmail.com(S. Suresh), gthamil@rediffmail.com(G. Thamizhendhi)

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Nonlocal conditions come up once values of the function on the boundary are connected to values within the domain. It is found to be a lot of plausible than the standard initial condi- tions for the formulation of some physical phenomena in certain problems of thermodynamics, elasticity tion x(0) and =
∑mwave propagation. In passing, we have a tendency to noticed that nonlocal condi-
k=1
ckx(tk) can be applied in physical problems yields better effect than the initial
conditions and the references therein [3, 5, 16, 17].

Motivated by the above works, to study an impulsive fractional integro differential equations with nonlocal condition of the form

cDθx(t) = U(t)x(t) + V (t) +

∫ t0 K(t, s)f(x(s))ds,0 <t<b, (1) x(t+k) = x(t−k) + yk,k = 1,2, ...m, yk ∈ X (2)

x(0) =
∑mk=1

ckx(tk), tk ∈ (0,b) (3)

where cDθ denotes the Caputo fractional derivative of order θ, 0 < θ < 1, f : X → X, K : {(t, s); 0 ≤ s ≤ t ≤ b} → R+, U, V : [0,b] → X are given appropriate functions, ck is real numbers and tk satisfy 0 = t0 < t1 < ... < tm < tm+1 = b.

The rest of this paper is planned as shades. In section 2, has definitions and elementary results of the fractional calculus. In section 3, the existence and uniqueness results for impulsive fractional integro differential equations involving nonlocal conditions are proved by using the standard fixed point theorems. In section 4, Some examples are illustrating the main results.

2 Preliminaries

Let us recall some basic definitions of fractional calculus. Let P = C([0,b],R) denote the Banach space of all continuous functions from [0,b] into R endowed with the usual norm defined by

x = sup{|x(t)|,t ∈ [0,b]}.

Definition 1. The fractional derivative of order θ > 0 of a function f : (0,∞) → X is given by
D0+θf(t) = Γ(n 1
− θ)(dtd)n ∫ t0
(t − f(s)
s)θ−n+1ds,
where n = [θ]+1, provided the right side is pointwise defined on (0,∞).

Definition 2. The fractional integral of order θ > 0 of a function f : (0,∞) → X is given by
I0+θf(t) = Γ(θ)
1∫ t0 (t − s)θ−1f(s)ds,
provided the right side is pointwise defined on (0,∞), where Γ(·) is the Gamma function.
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Definition 3. Let θ > 0 and f : [0,b] → X. The left sided Riemann-Liouville fractional integral of order θ of a function f is defined as
I0θ+f(t) = 1Γ(θ)
∫ t0 (t − s)θ−1f(s)ds, t ∈ [0,b],
Where f(t).
Γ(.) is the Euler gamma function and I0θ+f is exists for all α > 0. Moreover, I0θ+f(t) =

Definition 4. Let n − 1 <θ<n,n ∈ N and f ∈ Cn([0,b],X). The left side Caputo fractional derivative of order θ of a function f is defined as
cD0θ+f(t) = I0n−θ
+ dtdnnf(t). t ∈ [0,b],
Where n = [θ]+1, and [θ] denotes the integer part of the real number θ.

Lemma 1. Let 0 <θ<b, and V,f,K are continuous functions. If x ∈ C([0,b],X), then x satisfies the problem (1)-(2) if and only if u satisfies the integral equation:
x(t) =
A y1 ∑k=1 + mA
c∑kx(tΓ(θ) mk)
∫ tk
0 (tk − s)θ−1Hx(s)ds + Γ(θ)
1∫ t(t − s)θ−1Hx(s)ds, t ∈ [0,t1) 0 k=1
ckx(tk)
∫ Γ(θ)
0 tk
(tk − s)θ−1Hx(s)ds + 1Γ(θ)
y1 + y2 + A
∫ tk
∫ t(t − s)θ−1Hx(s)ds, t ∈ [t1,t2) 0 0 (tk − s)θ−1Hx(s)ds + Γ(θ)
1∫ t0 (t − s)θ−1Hx(s)ds, t ∈ [t2,t3) ...∑mi=0
∑mk=1
ckx(tk) Γ(θ)
yk + A
∑m∫ k=1 ckx(tk) Γ(θ)
tk (tk − s)θ−1Hx(s)ds + 0 1Γ(θ)
∫ t(t − s)θ−1Hx(s)ds, t ∈ (tm,b] 0 (4)
where
Hx(s) = U(s)x(s) + V (s) +
∫ s0 K(s, σ)f(x(σ))dσ and
A = 1 −
∑1 mk=1

ck

Theorem 1. (Krasnoselkii's fixed point theorem) Let K be a closed convex and nonempty subset of a banach space X.Let T and S, be two operators such that (i) $Tx + Sy \in K$ for any $x, y \in K$ (ii) T is compact and continuous. (iii) S is contraction mapping. Then there exists $z1 \in K$ such that $z1 = Tz1 + Sz1$.

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3 Main results

To prove the existence and uniqueness results we need the following assumptions :

- (A1) U(t) and V (t) are bounded and continuous function on [0,b].
- (A2) $f : X \to X$ is a continuous function.
- (A3) There exists constant $l > 0$ such that

$\|f(t, u) - f(t, u1)\| \le l\|u - u1\|, u, u1 \in X$

for each $t \in [0,b]$.

- (A4) $K : D \to R+$ is continuous on D with $K0 = \max\{|K(t, s)| : (t, s) \in D\}$, where $D = \{(t, s):0 \le s \le t \le b\}$.

Theorem 2. Assume that the assumption (A1),(A2), (A3) and (A4) are hold. If

∑mi=0

[A

ck(tk)θ + bθ]ρ Γ(θ + bK+ 1) 0l

< 1 (5)

then there exists a unique solution for the problem (1) − (3) on [0,b]

Proof:

We transform the problem (1)-(3) into a fixed point problem and define the operator $M : C([0,b],X) \to C([0,b],X)$ is given by

M(x)(t) =

||yk|| +

∑mk=1

∑myk + A ∑mi=0 k=1

ckx(tΓ(θ)

k)

∫ 0 tk

(tk − s)θ−1Hx(s)ds + 1Γ(θ)

∫ t(t − s)θ−1Hx(s)ds, t ∈ (tm,b] 0 (6)

where

Hx(s) = U(s)x(s) + V (s) +

∫ s0 K(s, σ)f(x(σ))dσ

A = 1 −

∑1 mk=1

ck

and define $Br = \{x \in C([0,b],X);\|x\| \le r\}$ for some $r > 0$. Choosing

r ≥

∑m||yk|| + 2[A ∑mck(tk)θ + bθk=0

k=1

](η + bK0µ0

)

Γ(θ + 1)

Let

µ0 = ||f(0)||, η = sup

||V (t)||, ρ = sup

||U(t)||. t∈[0,b]t∈[0,b]4

Step:1

We show that MBr ⊂ Br (i.e., the operator M has a fixed point on Br ⊂ C([0,b],X).

||M(x)(t)|| ≤

∑m∫ tk

(tk − s)θ−1||Hx(s)||ds i=0 0 + Γ(θ)

1||yk|| + A

$\sum m \int$ tk=1
ck$\|x(tk)\|$ $\Gamma(\theta)$
0 $(t - s)\theta-1\|Hx(s)\|ds$ (7)
where
$\|Hx(s)\| \leq \|U(s)\|\|x(s)\| + \|V(s)| +$
$\int$ s0 $\|K(s, \sigma)\|[\|f(x) - f(0)\| + \|f(0)\|]d\sigma \leq (\rho + bK0l)\|x\| + \eta + bK0\mu 0$
Therefore, the equation (8) we get,
$\|M(x)(t)\| \leq$
$\sum$mcki=0
$[(\rho + bK0l)r$
$\Gamma(\theta + 1) + \eta\ \Gamma(\theta + bK + 0\mu 1)$
0
$](tk)\theta$
+
$\|yk\| + A \sum m[(\rho + bK\Gamma(\theta +$ k=1
$0l)r\ 1) + \eta\ \Gamma(\theta + bK + 0\mu 1)$
0
$]b\theta$
$\leq$
$\sum$i=0 m$\|yk\| + \Gamma(\theta$ r
$+ 1)[A$
$ck(tk)\theta + b\theta][(\rho + bK0l) + \eta + bKr$
$0\mu 0$
]
$\leq r$
Step:2
Next we show that $M : Br \to Br$ is a contraction mapping. For any $u1,u2 \in Br$ and for $t \in (tm,b]$.
$\|Mu1(t) - Mu2(t)\| \leq$
$\sum$mk=1
$\sum m \int$ i=0
t0 $\|yk\| + A$
$\sum m(tk - s)\theta-1\|Hu1(s) - Hu2(s)\|ds$
$+ \Gamma(\theta)$
$1\int$ tk=1
ck $\Gamma(\theta)$
0 $(t - s)\theta-1\|Hu1(s) - Hu2(s)\|ds$
$\leq$
$\sum m[A\ ck(tk)\theta + b\theta$i=0
$]\rho\ \Gamma(\theta + bK + 1)0l$
$\|u1 - u2\|$
By (5), the operator M is a continuous. Hence by Banach's contraction principle, M has a unique fixed point which is a unique solution of the problem (1) − (3).
Theorem 3. Assume that the assumption (A1),(A2), (A3) and (A4) are hold. If
A
$\|yk\| +$
$\sum$mk=1
$\sum$mk=1
$ck[\rho + bK0l]\ \Gamma(\theta\ (tk) + \theta$
$1) < 1$ (8)
5
and
[A
$ck(tk)\theta + b\theta](\rho\ \Gamma(\theta + bK + 1)\ 0l)$
< 12 (9)
then there exists atleast one solution for the problem (1) − (3) on [0,b]
Proof:
We define the operator $M : C([0,b],X) \to C([0,b],X)$ is given by
$M(x)(t) =$

∑mk=1
∑m∑m∫ i=0
k=1
ckx(tΓ(θ)
k)
∫ 0 tk
(tk − s)θ−1Hx(s)ds + 1Γ(θ) tyk + A
(t − s)θ−1Hx(s)ds, t ∈ (tm,b] 0 (10)
where
Hx(s) = U(s)x(s) + V (s) +
∫ s0 K(s, σ)f(x(σ))dσ
A = 1 −
∑1 mk=1
ck
The operator M = M1 + M2 as follows
M1(x)(t) = A
∫ tk
0 (tk − s)θ−1Hx(s)ds (11)
M2(x)(t) =
∑mk=1
ckx(tk) Γ(θ)
∫ t0 (t − s)θ−1Hx(s)ds (12)
Now, we prove that M1x + M2x∗ ∈ Sr ⊂ C([0,b],X), for every x, x∗ ∈ Sr, Sr = {x ∈ C([0,b],X : ||x|| ≤ r}.
Let
µ0 = ||f(0)||, η = sup
t∈[0,b]||V (t)||, ρ = sup
t∈[0,b]||U(t)||.
r ≥
∑mi=0
yk + Γ(θ) 1∑m||yk|| + 2[A
∑mck(tk)θ + bθ) k=0
k=1
||M1x(t)|| ≤ A
](η + bK0µ0
Γ(θ + 1)
∑mck||x(tk)|| ∫ k=1 Γ(θ)
tk (tk − s)θ−1||Hx(s)||ds 0 (13)
where
||Hx(s)|| ≤ ||U(s)||||x(s)|| + ||V (s)| +
∫ s0 ||K(s, σ)||[||f(x) − f(0)|| + ||f(0)||]dσ ≤ (ρ + bK0l)||x|| + η + bK0µ0
6
Therefore, the equation (13) we get,
||M1x(t)|| ≤ A
ck[(ρ + bK0l)r
Γ(θ + 1) + η Γ(θ + bK+ 0µ1)
0
](tk)θ
||M2x∗ (t)|| =
∑mk=1
∫ t0 (t − s)θ−1||Hx*(s)||ds
||M2x∗ (t)|| ≤
∑mi=0
||yk|| + Γ(θ)
1[(ρ + bK0l)r
Γ(θ + 1) + η Γ(θ + bK+ 0µ1)
0
]bθ
≤
∑mi=0

||yk|| +
∑mi=0
||yk|| + Γ(θ r
+ 1)[A
∑mck(tk)θ + bθk=1
][(ρ + bK0l) + η + bKr
0μ0
]
Therefore,
||M1x + M2x∗ || ≤ ||M1x|| + ||M2x∗ || ≤ r
M1x + M2x∗ ∈ Sr Next, prove that the operator M1 is a contraction map on Sr and M2 is completely continuous on Sr.
||M1x(t) − M1x∗ (t)|| ≤ A
∑mk=1
ck(ρ + bK0l) Γ(θ (tk)+ θ
1)||x − x∗ ||
From (8), M1 is a contraction map on Sr.
Now we prove that (M2Sr) is uniformly bounded, (M2Sr) is equicontinuous and M2 : Sr → Sr is continuous. For any x ∈ Sr we have
||M2x(t)|| ≤
∑mi=0
[(ρ + bK0l)r
Γ(θ + 1) + (η Γ(θ + bK+ 0μ1)
0)
]bθ = l
Thus M2Sr ⊂ Sl and the set is uniformly bounded.
Let x ∈ Sr and t1,t2 ∈ [0,b] with t1 ≤ t2, we have
||M2x(t1) − M2x(t2)|| ≤
||yk|| +
∑m∫ i=0 ||yk|| + Γ(θ) 1+ Γ(θ) 1∫ t1
0 t2
(t2 − s)θ−1||Hx(s)||ds t1 (t1 − s)θ−1||Hx(s)||ds
≤
∑mi=0
||yk|| + 2((ρ + bK0l)r Γ(θ)
+ (η + bK0μ0)
)(t2 − t1)θ
We observe that, ||M2x(t1)−M2x(t2)|| → 0 when |t2−t1| → 0. Therefore, (M2Sr) is equicon- tinuous and M2 is completely continuous on Sr.
Hence by Arzela-Ascoli theorem, the operator M2 is compact on Sr. Therefore, the equation (1) − (3) has solution x(t) ∈ C([0,b],X. Hence the prove is completed.
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II. CONCLUSION

We study the existence of solutions of the initial value problem for impulsive fractional integro differential equations involving nonlocal conditions. The existence results are proved by using the fixed point theorems. Further, the problem (1)-(3) to study the existence of solutions for Caputo-Hadamard fractional integro differential equations involving fractional impulsive conditions.

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