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The Series of Geometric mean Matrix in Equalities

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ABSTRACT: *We discuss between the Heinz and logarithmic means .Weobtaine. sharp operator inequalities extending results given by Bhatia– Davis and Hiai–Kosaki on the series of the arithmetic-logarithmic-geometric mean matrix inequalities.[12]*

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I. INTRODUCTION.

There are several means that interpolate the geometric and series of arithmetic means; see [9], [13],[12] and [14]. One that attracts many researchers is the so-called Heinz mean $\sum_r H^r_{\varepsilon-1}$ given by

$$\sum_r H^r_{\varepsilon-1}(a_r, a_r+\varepsilon) = \sum_r \frac{{a_r}^{\varepsilon}(a_r+\varepsilon)^{\varepsilon+1} + {a_r}^{\varepsilon+1}(a_r+\varepsilon)^{\varepsilon}}{2}$$

Notice that $\sum_r H^r_0(a_r, a_r+\varepsilon) = H^r{}_1(a_r, a_r+\varepsilon) = \sum_r \frac{2a_r+\varepsilon}{2}$ is theseries of arithmetic mean and $\sum_r H^r_{\frac{1}{2}}(a_r, +\varepsilon) = \sum_r \sqrt{a_r}(a_r+\varepsilon)$ is the series of geometric mean.[12].

In 1951, Heinz [8], in his study of perturbation theory of operators, proved that for the operator norm $\|.\|$, given $A^{\frac{1}{2}}, B^{\frac{1}{2}}$ positive definite, for any X, that

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \frac{1}{2}\|A^{\varepsilon}XB^{1-\varepsilon} + A^{1-\varepsilon}XB^{\epsilon}\|. \tag{1}$$

In 1993, Bhatia–Davis [1] proved that if $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, and X are n by n matrices with $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$positivesemi definite, then for every unitarily invariant norm $|||.|||$ [12],

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \frac{1}{2}\|A^{\varepsilon}XB^{1-\varepsilon} + A^{1-\varepsilon}XB^{\epsilon}\| \le \frac{1}{2}\left\|A^{\frac{1}{2}}X + XB^{\frac{1}{2}}\right\| \tag{2}$$

Another mean, which is of interest mainly in chemical engineering, statistics, and thermodynamics, is theseries of logarithmic mean defined as

$$\sum_r L_r(a_r, a_r+\varepsilon) = \sum_r \frac{-\varepsilon}{\log a_r \log(a_r+\varepsilon)} = \sum_r \int_0^1 {a_r}^{\varepsilon}(a_r+\varepsilon)^{1-\varepsilon} d\varepsilon.$$

It is well known that

$$\sum_r G_r(a_r, a_r+\varepsilon) \le \sum_r L_r(a_r, a_r+\varepsilon) \le \sum_r A^{\frac{1}{2}}(a_r, a_r+\varepsilon) \tag{3}$$

In 1999, Hiai–Kosaki [10] obtained the following refinement of the inequality (2) showing:

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \left\|\left\|\int_0^1 A^{\varepsilon}XB^{\varepsilon-1}d\varepsilon\right\|\right\| \le \left\|A^{\frac{1}{2}}X + B^{\frac{1}{2}}X\right\| \tag{4}$$

called the series of arithmetic-logarithmic-geometric (A-L-G) inequality [12].

After seeing inequalities (2) and (4) it is hard not to be curious about the relationship between the Heinz andseries of logarithmic means. This was our motivation to investigate this problem[12].

Assume $\sum_r M_r(a_r, a_r+\varepsilon)$, $\sum_r N_r(a_r, a_r+\varepsilon)$ are symmetric homogeneous means on $(0,\infty) \times (0,\infty)$. $M_r$is said to strongly dominate $\sum_r N_r$ in notation$\sum_r M_r \ll \sum_r N_r$, if and only if the matrix$\sum_r \left[\frac{M_r(\lambda_i-\lambda_j)}{N_r(\lambda_i-\lambda_j}\right]_{i,j=1,\dots,n}$ is positive semidefinite for any $\lambda_1,...,\lambda_n > 0$ with any size n (see [11] for more details). Note that the inequality $\sum_r M_r \ll \sum_r N_r$ is stronger than theusual order $\sum_r M_r \le \sum_r N_r$. In [10], Hiai–Kosaki gave an example showing this. Another example was later obtained by Bhatia [4]. Moreover, if $A^{\frac{1}{2}}$ is a positive semidefinite matrix with eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$, then $\sum_r M_r \ll \sum_r N_r$ is equivalent to the operator norm inequality [12].

$$\sum_r |||M_r(A^{\frac{1}{2}}, A^{\frac{1}{2}})\circ X||| \le \sum_r |||N_r(A^{\frac{1}{2}}, A^{\frac{1}{2}})\circ X|||$$

where $\circ$ is the Schur–Hadamard or the entrywise product, and $\sum_r M_r(A^{\frac{1}{2}}, A^{\frac{1}{2}})$ is the matrix whose ij entry is $\sum_r M_r(\lambda_i,\lambda_j)$.[12]

Schur's theorem asserts that the Schur–Hadamard product of two positive matrices is positive. Two matrices $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ are said to be congruent if $B^{\frac{1}{2}} = S^* A^{\frac{1}{2}} S$ for some nonsingular matrix S. If $A^{\frac{1}{2}}$ is positive, then so is every matrix congruent to it. Aseries of a complex-valued function f_r on R is said to be positive definite if the matrix $[f_r((x_r)_i - (x_r)_j)]$ is positive semidefinite for all choices of points $\{(x_r)_1, (x_r)_2, \dots, (x_r)_n\} \subset R$ and all n = 1,2,.... Another interesting result that we are going to use is the well-known theorem of Bochner (see [11] for more details) which asserts that a series of function f_r in $L^1(R)$ is positive definite if and only if its Fourier transform $\hat{f_r}(\xi) \ge 0$, for almost all ξ. When calculating Fourier transforms, we ignore constant factors, since the only property of $\hat{f_r}$ we use is whether it is nonnegative almost everywhere.[12].

In this paper we first present a necessary and sufficient condition for the strong dominof the series of Heinz mean by theseries of logarithmic mean. This follows from the following theorem, which may be of independent interest, on the positive definiteness of functions; see [2], [3], [4], [5], [6], and [11] for other results on positive definiteness of functions. Second, using a standard result on a norm of the Schur multiplier, we derive norm inequalities extending results given by Bhatia–Davis and Hiai–Kosaki on A-L-G mean matrix inequalities.[12].

Theorem 1. Let $\sum_r f_r(x_r) = \sum_r \frac{x_r \cosh((\varepsilon-1)x_r)}{\sinh(x_r)}$.

Then f_r is positive definite if and only if

$$-\frac{3}{2} \le \varepsilon \le \frac{3}{2}$$

The following formulas are known from [7] and we provide the proofs for completeness

Lemma 1. For $\varepsilon \le 0$, we have

$$\int_0^\infty \frac{\sinh(1-\varepsilon)x_r}{\sinh(x_r)} \cos(1+\varepsilon)x_r \, dx_r = \frac{\pi \sin(1-\varepsilon)\pi}{2(\cosh((1+\varepsilon)\pi + \cos((1-\varepsilon)\pi)} \qquad (5)$$

$$\int_0^\infty \frac{\cosh(1-\varepsilon)x_r}{\sinh(x_r)} \sin(1+\varepsilon)x_r \, dx_r = \frac{\pi \sinh(1+\varepsilon)\pi}{2(\cosh((1+\varepsilon)\pi + \cos((1-\varepsilon)\pi)} \qquad (6)$$

Proof. To compute the above integrals we use the method of residues. We proceed in two steps.

Step 1. Let us consider the complex valued function

$$\sum_r \varphi_r(z_r) = \sum_r \frac{\sinh((1-\varepsilon)z_r)}{\sinh(z_r)} e^{i(1+\varepsilon)z_r}$$

Then φ_r has poles at the points $z_k = ik\pi$, for $k = \pm 1, \pm 2, \dots$ Now, consider the contour integral $\sum_r \int_\Gamma \varphi_r(z_r) dz_r$, where Γ is the rectangle with vertices at $(-R, 0), (R, 0), (R, i\pi)$ and $(-R, i\pi)$ described counterclockwise, with an indentation $\gamma_\varepsilon : z_r = \varepsilon e^{i\theta}$ for $0 \ge \theta \ge -\pi$, so as to avoid the pole at $i\pi$. Since there are no singularities of the integrand inside Γ, we obtain by Cauchy's theorem for analytic functions

$$\int_{-R}^{R} \sum_r \varphi_r(x_r)\, dx + \int_0^\pi \sum_r \varphi_r(R + iy_r) i dy_r$$
$$+ \int_R^\varepsilon \sum_r \varphi_r(x_r + i\pi) dx$$
$$+ \int_{\gamma_\varepsilon} \sum_r \varphi_r(z_r)\, dz_r + \int_{-\varepsilon}^{R} \sum_r \varphi_r(x_r + i\pi) dx_r + \int_\pi^0 \sum_r \varphi_r(-R + iy_r) i\, dy_r.$$

Using the estimation lemma, we obtain along the two vertical lines

$$\left|\int_0^{\pi}\sum_r \varphi_r\,(R+iy_r)idy_r\right| \to 0 \;\text{ and }\; \left|\int_{\pi}^{0}\sum_r \varphi_r\,(-R+iy_r)idy_r\right| \to 0 \text{ as } R\to\infty\,. \qquad (7)$$

By Jordan's lemma, we get

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\sum_r \varphi_r\,(z_r)dz_r = i(-\pi-0)(-i\sin(1-\varepsilon)\pi)e^{(1+\varepsilon)\pi} \qquad (8)$$

On the other hand, using the identities

$$\sinh(a\pm ib) = \sinh(a)\cos(b) \pm i\cosh(a)\sin(b),$$

we obtain

$$\int_R^{\varepsilon}\sum_r \varphi_r\,(x_r+i\pi)dx = \sum_r e^{-(1+\varepsilon)\pi}\int_{\varepsilon}^{R}\frac{e^{i(1+\varepsilon)x}}{\sinh(x_r)}[\sinh(1-\varepsilon)\pi + i\cosh(1-\varepsilon)\pi\sin(1-\varepsilon)\pi]\,dx$$

and

$$\int_{-\varepsilon}^{R}\sum_r \varphi_r\,(x_r+i\pi)dx = \sum_r e^{-(1+\varepsilon)\pi}\int_{\varepsilon}^{R}\frac{e^{-i(1+\varepsilon)x}}{\sinh(x_r)}[\sinh(1-\varepsilon)\pi - i\cosh(1-\varepsilon)\pi\sin(1-\varepsilon)\pi]\,dx.$$

Combining the two above identities and using Euler's formula, we obtain aftersimplifications [12]

$$\int_R^{\varepsilon}\sum_r \varphi_r\,(x_r+i\pi)dx_r$$

$$+\sum_r\int_{-\varepsilon}^{-R}\sum_r \varphi_r\,(x_r+i\pi)dx_r$$

$$=\sum_r e^{-i(1+\varepsilon)x_r}\left\{\cos(1-\varepsilon)\pi\int_{\varepsilon}^{R}\frac{\sinh(1-\varepsilon)\pi}{\sinh x_r}\,(2\cos(1+\varepsilon)\;dx_r\right.$$

$$\left.+\,i\sin(1-\varepsilon)x_r\int_{\varepsilon}^{R}\frac{\cosh(1-\varepsilon)x_r}{\sinh x_r}\,(2i\sin(1+\varepsilon)x_r\,dx_r\right\}.$$

Using

$$\int_{-R}^{R}\sum_r \varphi_r\,(x_r)dx_r = \sum_r\int_{-R}^{R}\frac{\sinh(1-\varepsilon)\pi}{\sinh x_r}\cos(1+\varepsilon)\;dx_r = 2\sum_r\int_0^{R}\frac{\sinh(1-\varepsilon)\pi}{\sinh x_r}\cos(1+\varepsilon)\;dx_r$$

and taking $\varepsilon\to 0$ then after that $R\to\infty$, we obtain

$$2\int_0^{\infty}\frac{\sinh(1-\varepsilon)\pi}{\sinh x_r}\cos(1+\varepsilon)\;dx_r$$

$$+\sum_r e^{-(1+\varepsilon)\pi}\left\{2\cos(1-\varepsilon)\,\pi\int_0^{\infty}\frac{\sinh(1-\varepsilon)\pi}{\sinh x_r}\cos(1+\varepsilon)\;dx_r - 2\sin(1-\varepsilon)\pi\right.$$

$$\left.\times\int_0^{\infty}\frac{\cosh(1-\varepsilon)\pi}{\sinh x}\sin(1+\varepsilon)x_r\,dx_r - \pi\sin(1-\varepsilon)\,\pi\right\} = 0. \qquad (9)$$

Step 2. Similarly as in Step 1, we may consider the complex valued function

$$\sum_r \Psi_r(z_r) = \sum_r\frac{\cosh(1-\varepsilon)\,z_r}{\sinh z_r}e^{i(1+\varepsilon)z_r}$$

Then Ψ_r has poles at $z_k = \pm ik\pi$ *where* $k = 0,1,2,\ldots$Consider the contour integral $\int_\Gamma\sum_r\Psi_r(z_r)dz$, where Γ is the same contour as in Step 1 with two indentations $\gamma_{\varepsilon_1}: z=\varepsilon e^{i\theta}+i\pi$ for $0\geq\theta\geq-\pi$, *so* as to avoid the pole at $i\pi$, and $\gamma_{\varepsilon_2}: z=\varepsilon e^{i\theta}$ *for* $0\geq\theta\geq-\pi$, so as to avoid the pole at 0. By applying Cauchy's theorem, .[12]we obtain

$$\int_{R}^{\varepsilon_2}\sum_r \Psi_r(x_r)\,dx + \int_{\gamma_{\varepsilon_2}}\sum_r \Psi_r(z_r)\;dz + \int_{\varepsilon_2}^{R}\sum_r \Psi_r(x_r)\,dx + \int_{0}^{\pi}\sum_r \Psi_r\,(R+iy_r)idy_r$$
$$+\int_{R}^{\varepsilon_1}\sum_r \Psi_r\,(x_r+i\pi)dx + \int_{\gamma_{\varepsilon_1}}\sum_r \Psi_r(z_r)\;dz_r + \int_{-\varepsilon_1}^{-R}\sum_r \Psi_r\,(x_r+i\pi)dx_r$$
$$+\int_{\pi}^{0}\sum_r \Psi_r\,(-R+iy_r)idy_r = 0.$$

By Jordan's lemma, we get in Step 1

$$\lim_{\varepsilon_1\to 0}\int_{\gamma_{\varepsilon_1}}\sum_r \Psi_r(z_r)\;dz_r = i\,(-\pi-0)(-cos(1-\varepsilon)\pi e^{-(1+\varepsilon)\pi} = i\pi cos(1-\varepsilon)e^{-(1+\varepsilon)\pi}$$

and

$$\lim_{\varepsilon_2\to 0}\int_{\gamma_{\varepsilon_2}}\sum_r \Psi_r(z_r)\;dz = i\,(-\pi-0)(-cosh(0)e^{0} = i\pi\,.$$

After similar arguments as in Step 1, with some small changes, by taking limits as$\varepsilon_2\to 0, \varepsilon_1\to 0$ and $R\to\infty$, successively,.[12] we get

$$2i\sum_r\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} sin(1+\varepsilon)\,x_r dx_r$$
$$+\sum_r e^{-(1+\varepsilon)\pi}\Bigg\{2icos(1-\varepsilon)\pi\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} sin(1+\varepsilon)\,x_r dx_r$$
$$+\sum_r 2i\,sin(1-\varepsilon)\,x_r\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} cos(1+\varepsilon)\,x_r dx_r + i\pi\,cos(1-\varepsilon)\Bigg\} - i\pi\; = 0 \quad (10)$$

Let$I=\sum_r\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} cos(1+\varepsilon)\,x_r dx_r$ and $J=\sum_r\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} sin(1+\varepsilon)\,x_r dx_r$ Then (9) and (10) can be written, successively, as

$$\begin{cases}\left(2+2e^{-(1+\varepsilon)\pi}\,cos(1-\varepsilon)\right)I - 2e^{-(1+\varepsilon)\pi}\,sin(1-\varepsilon)\,J - \pi\,sin(1-\varepsilon)e^{-(1+\varepsilon)\pi} = 0\\ \left(2+2e^{-(1+\varepsilon)\pi}\,cos(1-\varepsilon)\right)J + 2e^{-(1+\varepsilon)\pi}sin(1-\varepsilon)I + \pi cos(1-\varepsilon)e^{-(1+\varepsilon)\pi} - \pi = 0.\end{cases}$$

Solving the above system for I and J we obtain the desired results.

Proof of Theorem 1. Using Bochner's theorem, the positive definiteness of the function f_r can be reduced to showing that the Fourier transform $\sum_r \hat{f}_r\,(1+\varepsilon)$ is positive.[12]. Since f_r is an even function, its Fourier transform is given by

$$\sum_r \hat{f}_r\,(1+\varepsilon) = 2\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} cos(1+\varepsilon)\,x_r dx_r$$

The differentiation of the formula (5) in Lemma 1 with respect to $1-\varepsilon$ gives

$$\int_0^\infty \frac{cosh(1-\varepsilon)\pi}{sinh\,x_r} cos(1+\varepsilon)\,x_r dx_r = \sum_r \frac{\pi}{2}\frac{\pi\,cos(1-\varepsilon)\,x_r[cosh(1+\varepsilon)\,\pi + cos(1-\varepsilon)\,\pi] - sin(1-\varepsilon)\,\pi}{(cosh(1+\varepsilon)\,\pi + cos(1-\varepsilon)\,\pi)^2}$$
$$= \frac{\pi^2[1+cos(1-\varepsilon)\pi cosh(1+\varepsilon)\pi}{2(cosh(1+\varepsilon)+cos(1-\varepsilon)\pi)^2}$$

So,

$\sum_r \hat{f}_r\,(1+\varepsilon) = \frac{\pi^2[1+cos(1-\varepsilon)\pi cos\,h(1+\varepsilon)\pi}{(cos\,h(1+\varepsilon)+cos(1-\varepsilon)\pi)^2}$.

Consequently, if $1\le\varepsilon\le\frac{3}{2}0$, then $\sum_r \hat{f}_r\,(1+\varepsilon)\;\ge\;0$. Since φ_r is even in$1+\varepsilon$, the result follows for-$\frac{3}{2}\le\varepsilon\le\frac{3}{2}$.

Corollary 1. .[12]For any $a,b\;\ge\;0$, we have

$$\sum_r H_\varepsilon{}^r(a,a+\varepsilon) << \sum_r L_r(a_r,a_r+\varepsilon)\;if\;and\;only\;if\;.\frac{1}{4}\le\varepsilon\le\frac{3}{4}. \quad (11)$$

Corollary 2. .[12]Let $A^{\frac{1}{2}}, B^{\frac{1}{2}}$ be any positive matrices. Then for any matrix X and for $\frac{1}{4}\le\varepsilon\le\frac{3}{4}$,, we have

$$\|A^{1-\epsilon}XB^{\epsilon}+A^{\epsilon}XB^{1-\epsilon}\| \le 2\left|\left|\left|\int_0^1 A^{\epsilon}XB^{1-\epsilon}d\epsilon\right|\right|\right| \qquad (12)$$

for every unitarily norm |||. |||
Proof of Corollary 1.[12] We proceed in two steps.
Step 1. By definition, $\sum_r H_\varepsilon^r(a, a+\epsilon) << \sum_r L_r(a, a+\epsilon)$ *if*

$$(1-\epsilon)_{ij} = \sum_r \left[\frac{H_\varepsilon^r(\lambda_i,\lambda_j)}{L_r(\lambda_i,\lambda_j)}\right]_{i,j=1,\dots,n}$$

is positive semidefinite. Put$\lambda_i = e^{x_i}, \lambda_j = e^{x_j}$, *with* $x_i, x_j \in R$ .Then

$$(1-\epsilon)_{ij} = \left(\frac{e^{(1-\varepsilon)(\frac{x_i-x_j}{2})}+e^{(1-\varepsilon)(\frac{x_j-x_i}{2})}}{e^{\frac{x_i}{2}}(\frac{e^{(\frac{x_i-x_j}{2})}-e^{(\frac{x_j-x_i}{2})}}{x_i-x_j})e^{\frac{x_j}{2}}}\right).$$

Thus the matrix $[(1-\varepsilon)_{ij}]$ is congruent to one with entries

$$\frac{\left(\frac{x_i-x_j}{2}\right)cosh(1-\varepsilon)\left(\frac{x_i-x_j}{2}\right)}{sinh\left(\frac{x_i-x_j}{2}\right)}$$

where $\varepsilon = 1$. Hence, the matrix $[(1-\varepsilon)_{ij}]$is positive semidefinite if and only if the function

$$\sum_r f_r(x_r) = \sum_r \frac{x_r cosh(1-\varepsilon)x_r}{sinhx_r}$$

is positive definite.
Step 2. By Theorem 1, $f_r(x)$ is positive definite if and only if$\frac{1}{2} \le \varepsilon \le \frac{3}{2}$, which is equivalent to the condition $\frac{1}{4} \le \varepsilon \le \frac{3}{4}$.
Remark 1. .[12]The inequality $M_r < N_r$ could, in general, be strictly stronger than the usual inequality $M_r \le N_r$. That means not every inequality between means of positive numbers leads to a corresponding inequality for positive matrices as shown by the following simple example. For $a_r > 0$ we have

$$\sum_r H^r{}_{1-\varepsilon}(a_r, a_r+\varepsilon) \le \sum_r L_r(a_r, a_r+\varepsilon) \text{ if and only if} \frac{1-\frac{1}{\sqrt{3}}}{2} \ge \varepsilon \ge \frac{1+\frac{1}{\sqrt{3}}}{2}. \qquad (13)$$

In fact, by taking $a_r = e^x$and $(a_r+\varepsilon) = e^{y_r}$and using Taylor series, it is easy to see that $\sum_r H^r{}_{1-\varepsilon}(a_r, a_r+\varepsilon) \le \sum_r L_r(a_r, a_r+\varepsilon)$if and only if

$$cosh\left((1-2\varepsilon)\left(\frac{x_r-y_r}{2}\right)\right) \le \frac{sinh\left(\frac{x_r-y_r}{2}\right)}{\left(\frac{x_r-y_r}{2}\right)}.$$

Let$\varepsilon = \frac{x_r-y_r}{2}$,and $\varepsilon = 1$. Then after simplification

$$1+\frac{(1-\varepsilon)^2\varepsilon^2}{2!}+\frac{(1-\varepsilon)^4\varepsilon^4}{4!}+\cdots \le 1+\frac{\varepsilon^2}{3!}+\frac{\varepsilon^4}{5!}+\cdots$$

This is true only if$(1-\varepsilon)^2 \le \frac{1}{3}$, which leads to the desired result.

Proof of Corollary 2. First assume $A^{\frac{1}{2}} = B^{\frac{1}{2}}$Since the norms involved are unitarily invariant, we may suppose that $A^{\frac{1}{2}}$ is diagonal with entries$\lambda_1, \lambda_2, \dots, \lambda_n$. Then we have

$$A^{1-\varepsilon}XA^{\varepsilon}+A^{\varepsilon}XA^{1-\varepsilon} = Yo\left(\int_0^1 A^{\varepsilon}XA^{1-\varepsilon}\,d\varepsilon\right)$$

where Y is the matrix with entries

$$(y_r)_{ij} = \sum_r \frac{2H^r{}_{1-\varepsilon}(\lambda_i,\lambda_j)}{L_r(\lambda_i,\lambda_j}.$$

A well-known result on the Schur multiplier norm (see [12, Theorem 5.5.18 and Theorem 5.5.19]) says that if Y is any positive semidefinite matrix, then for all matrix X,.[12]

$$|||Y\ o\ X||| \le \max_i\{y_{ii}|||X|||, \text{for every unitarily invariant norm.} \qquad (14)$$

By Corollary 1, Y is a positive semidefinite matrix. Applying (14), .[12] we obtain

$$|||A^{1-\varepsilon}XA^{\varepsilon}+A^{\varepsilon}XA^{1-\varepsilon}||| \le 2\left|\left|\left|\int_0^1 A^{\varepsilon}XA^{1-\varepsilon}\,d\varepsilon\right|\right|\right| \qquad (15)$$

Now, we use the usual trick replacing $A^{\frac{1}{2}}$ and X in the inequality (15) by the 2 by 2 matrices $\begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. This gives us the desired inequality (12).

Remark 2. Given $a, \varepsilon > 0$. A natural question arises as to whether the reverse inequality $L(a, a+\varepsilon) << H_{1-\varepsilon}(a, a+\varepsilon)$ is valid.[12]

For $\varepsilon = 0{,}1$ we have $\sum_r L_r(a_r, a_r+\varepsilon) << \sum_r {H^r}_{1-\varepsilon}(a_r, a_r+\varepsilon)$ (which is exactly the second part of On the other hand,

cannot be true for $\epsilon \in (0,1)$ due to the fact that $\sum_r f_r(x) = \sum_r \frac{\sin h x_r}{x_r \cos h(1-2\varepsilon)}$ goes to infinity as $x_r \to \pm\infty$. So, f cannot be positive definite.[12].

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