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Investigation of the Efficiency of the Solution of a Simple Mechanical Model by Using Laplace Transformation

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ABSTRACT:*The goal of this study is to solve differential equations governing a simple mechanical system by transforming it into the frequency domain and then evaluate the effectiveness of the method used as a result of the comparison between the solutions that are transformed to the time domain and the analytical solutions. For this purpose, firstly analytical Laplace and FFT based numerical Laplace transformation methods are compared using an earthquake acceleration data. Even though the values obtained from these two methods overlap, the FFT based direct Laplace transform is observed to be more appropriate because of its lower computing time requirement. In addition to that, Durbin's inverse transform method and its modified version are compared numerically. It has been shown that the Durbin's modified inverse transform method gives results that are more effective. Finally, solutions for a problem from a simple mechanical model obtained in the frequency domain are compared with analytical solutions. The numerical results of this study show that solutions of time-dependent differential equations in the Laplace domain overlap with the analytical results.*

KEYWORDS: *Frequency domain, Laplace transform, Inverse Laplace transform.*

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I. INTRODUCTION

Dynamic behavior of structures is expressed by ordinary or partial differential equations. In order to evaluate the dynamic behavior of structures, these time-dependent differential equations must be solved. Time-dependent differential equations can be solved by direct numerical integration methods. In direct numerical integration methods, the solution interval is divided into small time increments. Another approach for the solution of differential equations governing system behavior is to carry out the solution in the frequency domain.

Time dependent differential equations can be solved by the Fourier transform approach, which is one of numerical operational methods. Huang and Yang [1] simulated dynamic analysis of a rotating beam. They have used Frequency-domain responses for examining the dynamic characteristics of the beam in their studies. Lee and Oh [2] derived a spectral element model for the dynamics and stability analyses of beams subject to axial tension. The authors have computed the dynamic responses in frequency domains. Park et al. [3] developed a spectral element model to represent the dynamic response of a coupled piezoelectric wafer and beam system. They have demonstrated spectral element model with numerical examples in time and frequency domains. Kim and McCullough [4] investigated the dynamic displacement and stress responses of a plate on a viscous Winkler foundation in the transformed field domain.

Time dependent linear differential equations can also be solved efficiently in the Laplace domain. In this approach, applying Laplace transform to the time-dependent differential equations removes the time-dependency and the resulting linear equations can easily be solved by numerical methods in the Laplace domain. The solutions in the Laplace domain are then transformed accurately to the time domain with the method such as modified Durbin's numerical inverse Laplace transform method [5]-[8]. Folch et al. [9] have solved the problems with vıscoelastic materials using Prony series and inverse Laplace transform. Gaul et al. [10] have investigated the dynamıc behaviors of viscoelastic solids by boundary element method based on inverse Laplace transform. Massouros et al. [11] have conducted studies about Laplace transform, in particular on numerical inverse Laplace. In addition, De Chant [12] has discussed the limitations of inverse Laplace transform.

In this study, firstly the second-order differential equation of the simple mechanical model subjected to impulsive load is obtained in the frequency domain with the help of Laplace transform. And this equation that is transformed to the frequency-domain is solved and then these solutions are transformed back to the time-domain using the Durbin's modified inverse Laplace transform. Numerical results are then compared with those of analytical benchmark solutions of the problem. It has been shown that these transformed solutions overlap with the analytical benchmark solutions

II. LAPLACE AND INVERSE LAPLACE TRANSFORM APROACH

2.1. Laplace Transform

In dynamic problems, the Laplace transformation of a function with respect to time can be done in various different ways. The Laplace transformation of some functions with known analytical forms can be done in closed form. The Laplace transform $\overline{F}(s)$ of a function *f*(*t*) with respect to time is defined as,

$$L[f(t)] = \bar{F}(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{1}$$

where *s*, is the Laplace transform parameter and it is in general a complex number. It is very easy to show that

$$L[\dot{f}(t)] = s\bar{F}(s) - f(0) \tag{2}$$

$$L[\ddot{f}(t)] = s^2\bar{F}(s) - sf(0) - \dot{f}(0) \tag{3}$$

where $f(0)$ and $\dot{f}(0)$ are initial conditions at *t*=0.

In situation where experimental data or data from earthquake records is available, the Laplace transform can be done analytically with the assumption that the function is linear between two points. Laplace transform of function *f*(*t*) with respect to time and linearly changing in the two time intervals is defined as

$$\bar{F}(s) = \sum_{i=0}^{N-1} \frac{1}{s^2\,\Delta t}\left[s\,\Delta t\,\{f(t_i)e^{-st_i} - f(t_{i+1})e^{-st_{i+1}}\} + \Delta f\{e^{-st_i} - e^{-st_{i+1}}\}\right] \tag{4}$$

where $\Delta f = f(t_{i+1}) - f(t_i)$ and $\Delta t = t_{i+1} - t_i$. In Eq. (4), summations have been repeated for various Laplace parameters (*s*).

The analytical forms of some functions are complex and hence the Laplace transform cannot be done in closed form. In these cases, discrete values of function are calculated and then transformation can be done by direct Laplace transform based on Fast Fourier Transform (FFT).

Function *f*(*t*) with respect to time is defined as,

$$f(t) = f(t) \quad \Rightarrow \quad 0 \le t \le T$$

$$f(t) = 0 \quad \Rightarrow \quad t < 0 \text{ and } t > T$$

Laplace transform$\overline{F}(s)$ is defined as,

$$\bar{F}(s) = \int_0^T f(t)e^{-st}\,dt \tag{5}$$

For the discrete values of *f*(*t*) at the $t=t_n=n\Delta t$ points can be shown as follows.

$$\bar{F}(s_k) = \Delta t \sum_{n=0}^{N-1} [f(t_n)e^{-at_n}]\, e^{-i\frac{2\pi nk}{N}} \tag{6}$$

in which $s_k = a + ik2\pi/T$, $N = T/\Delta t$ where s_k is the k*th* Laplace transform parameter, *T* is the solution interval and Δt is the time increment. The discrete Laplace transform in Eq. (6) is also done by using the FFT sub-program.

2.2. Numerical Inverse Laplace Transform

Inverse Laplace transform is defined as

$$L^{-1}[\bar{F}(s)] = f(t) = \frac{1}{2\,\pi\,i}\int_{a-i\infty}^{a+i\infty} \bar{F}(s)e^{st}\,ds \tag{7}$$

where *a* is an arbitrary positive constant that eliminates singularity in all real parts of $\bar{F}(s)$.

In physical problem analysis, an effective numerical discrete inverse transformation method is needed to transform from the solution in Laplace domain to the time domain. From these inverse transform methods, Durbin's inverse transform is the most important. This method is the improved version of the Dubner and Abate [13] method which is based on Finite Fourier-Cosine transform.

Durbin's numerical inverse Laplace transform [14] is given by,

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T}\left[-\frac{1}{2}Re\{\bar{F}(a)\} + Re\left\{\sum_{k=0}^{N-1}\left(A(k) + iB(k)\right)e^{\left(i\frac{2\pi}{N}\right)jk}\right\}\right] \tag{8}$$

$$A(k) = \sum_{l=0}^{L} Re\left\{\bar{F}\left(a + i\,(k + l\,N)\frac{2\pi}{T}\right)\right\} \tag{9}$$

$$B(k) = \sum_{l=0}^{L} Im\left\{\bar{F}\left(a + i\,(k + l\,N)\frac{2\pi}{T}\right)\right\} \tag{10}$$

where, i is the complex number, T is sampling time interval, N is the total number of equidistant sampling points (N=2m : m being integer), $s_k = a + ik2\pi/T$ is kth Laplace transform parameter, $tj=j\Delta t=jT/N$, (j=0,1,2,….N-1). In Eq. (8), the second part of the equality between the brackets is

$$\left\{\sum_{k=0}^{N-1}\left(A(k) + i\,B(k)\right)e^{\left(i\frac{2\pi}{N}\right)jk}\right\} \tag{11}$$

calculated by using a Fast Fourier Transform sub-program [15]. Eq. (8) can also be modified according to the Narayanan's suggestion [16].

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T}\left[-\frac{1}{2}Re\{\bar{F}(a)\} + Re\left\{\sum_{k=0}^{N-1}(\bar{F}(s_k)L_k)e^{\left(i\frac{2\pi}{N}\right)jk}\right\}\right] \tag{12}$$

where, each term of discrete values that is calculated in the Laplace domain is modified by multiplying them with Lanczos (L_k) factor. These factors are given by:

$$L_k \quad : \quad \begin{cases} = 1 & , \quad k = 0 \\ = Sin\left(\frac{k\pi}{N}\right)\Big/\left(\frac{k\pi}{N}\right) & , \quad k > 0 \end{cases} \tag{13}$$

It should be noted that, the selection of the appropriate values of parameters N, a and T are critical in order to achieve the desired accuracy in the inverse transform. In the literature it is indicated that setting the value of T and choosing the value of a multiplied by T (aT) in between $5 \leq aT \leq 10$ yields the value of a necessary for the required precision [14]. For the numerical examples presented in this paper the value of 'aT' is taken as '6'.

III. NUMERICAL EXAMPLES AND DISCUSSION

Example 1. Comparison of Solution of Analytical Laplace and Direct Laplace Transform

The objective of this example is to compare the precisions of the two methods, namely analytical Laplace and numerical direct Laplace based on FFT. Towards this goal, the north-south component of the 1940 El-Centro earthquake acceleration data is considered. N=2048 parameters have been used for numerical direct Laplace and analytical Laplace transform. In the following Laplace transform $s_k=a+i(2\pi k/T)$, a and T values are selected as 0.15 and 40.96 seconds respectively. The real and imaginary parts of the transform are compared in Figure 1-2.

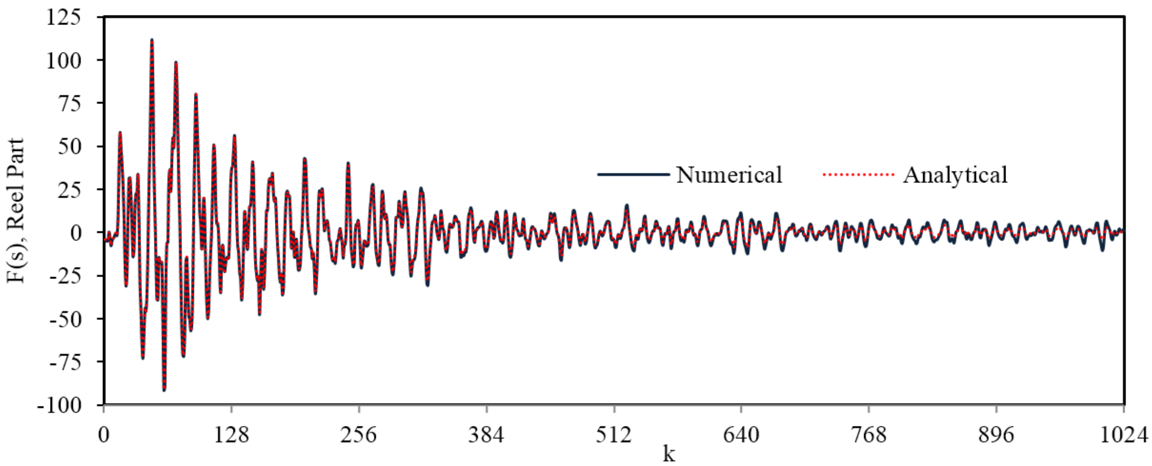

Figure 1. Real part of Laplace transformation of El-Centro earthquake record.

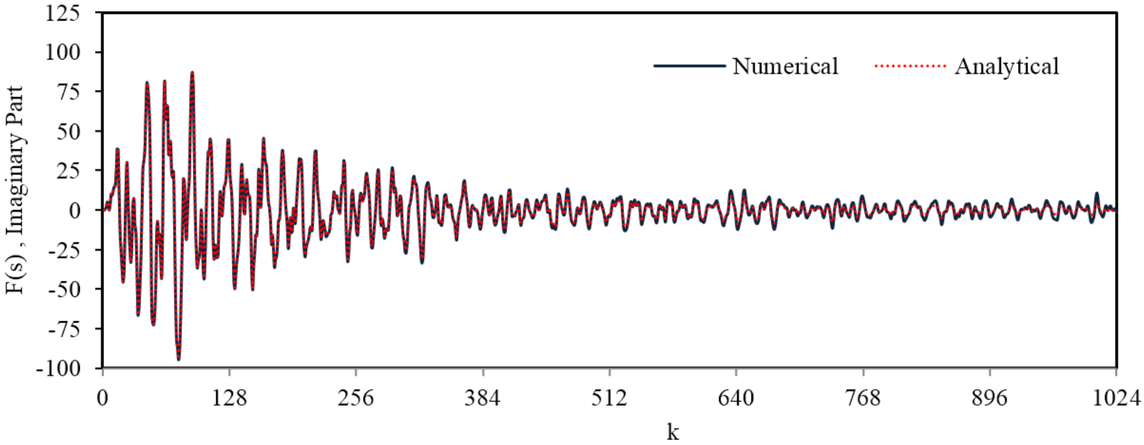

Figure 2. Imaginary part of Laplace transformation of El-Centro earthquake record.

When the Figures 1 and 2 are examined, it is seen that the results of Laplace transform using FFT are very close to the results of Analytical Laplace transformation. Since Analytical Laplace transformation requires a lot of computing time, the numerical direct Laplace transformation method based on FFT can be used more effectively.

Example 2. Comparison of Durbin and Modified Durbin Inverse Laplace Transform

In this example, the goal is to show the precisions of Durbin's inverse transform method and its modified version. For this, some load functions with known closed Laplace transform given in Table 1 are used.

Table 1. Laplace transform of some time dependent load function

Load Type	Load Function, *f(t)*	Laplace Transform, $\overline{F}(s)$
Periodic Rectangular	$2\sum_{k=0}^{\infty}(-1)^k U(t-2k)$	$\frac{2}{s(1+e^{-2s})}$
Heaviside Unit Function	U(t-25)	$\frac{e^{-25s}}{s}$
Increased Sine Function	t/2(Sin t)	$\frac{s}{(s^2+1)^2}$

The values in Laplace domain are obtained using the closed Laplace transforms of these functions and the inverse transforms of these values are realized with Durbin and the modified Durbin methods. Then these values transformed back to the time domain are compared with the analytical values as shown in Figure 3-5. In the inverse transformation aT, N and L values are selected as 6, 256 and 4, respectively.

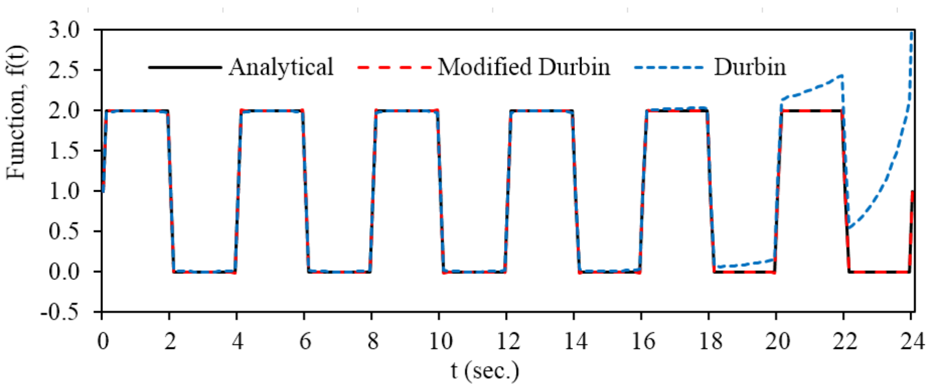

Fig. 3. Time change of periodic rectangle function

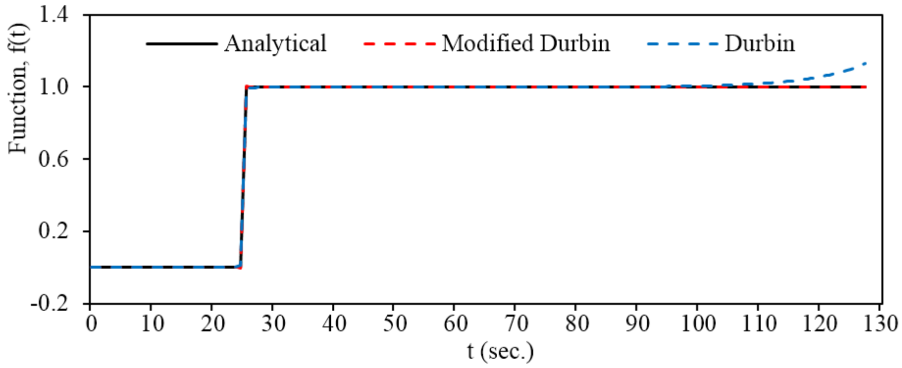

Fig. 4. Time change of Heaviside Unit function

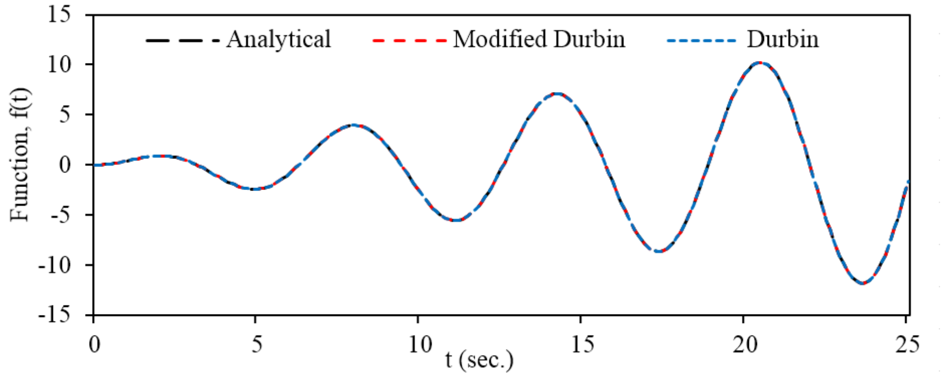

Fig. 5. Time change of Increased Sine function

As seen in Figures 3-5, the results obtained from Durbin's Modified Method overlap with the analytical results.

Example 3. A Simple Mechanical Model Subjected to a Sinusoidal Load

In this example, a single degree of freedom mechanical model subjected the sinusoidal load seen in Fig. 6. has been considered and a differential equations governing this model is obtained. The solutions of the differential equations governing the mechanical system have been obtained in the frequency domain by using Laplace transform

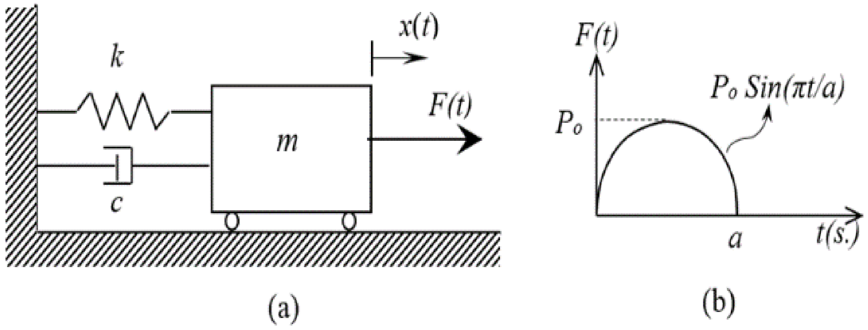

Fig. 6. A simple mechanical model. (a) Mechanical Model, (b) Dynamic load

Initially, the Laplace transforms are applied for the differential equations governing the behavior of the mechanical model. Time independent equation of motion is solved in the Laplace domain and then the time domain solutions of them are found by reversing with the inverse transforms. The obtained numerical results are compared with the analytical solution of the problem.

The normalized equation of motion governing the behavior of the system can be expressed as follows:

$$\ddot{x} + 2\omega^*\xi\,\dot{x} + (\omega^*)^2 x = \frac{F(t)}{m} \tag{14}$$

wherein, represents the displacement, and it depends on time for the dynamic problems. $(\omega^*)^2$ is describedby k/m. Here, k and m represent the spring constant and the mass of the system, respectively. $\xi = c/2m\omega^*$ is the dimensionless damping ratio and c is the damping constant. F (t), dynamic load function acting on the system is given below

$$F(t) = P_0 \,\mathrm{Sin}\left(\frac{\pi\, t}{a}\right) \tag{15}$$

First, the Laplace transform of the governing equation of motion (14) is written as:

$$s^2\bar{x}(s) - s\,x(0) - \dot{x}(0) + 2\xi\omega^*[s\,\bar{x}(s) - x(0)] + (\omega^*)^2\bar{x}(s) = \frac{\bar{F}(s)}{m} \tag{16}$$

here, s is the Laplace parameter, $x(0)$ and $\dot{x}(0)$ indicatethe initialdisplacementandvelocity, respectively. Eq. (16),under zero initial conditions takes the form:

$$s^2\bar{x}(s) + 2\xi\omega^*\,s\,\bar{x}(s) + (\omega^*)^2\bar{x}(s) = \frac{\bar{F}(s)}{m} \tag{17}$$

Arranging the Eq. (17), displacement expression is obtained in the Laplace domain as:

$$\bar{x}(s) = \frac{1}{m}\left(\frac{1}{s^2 + 2\xi\omega^*\,s + (\omega^*)^2}\right)\bar{F}(s) \tag{18}$$

When, a=π is selected for the applied sinusoidal impulsive load, the Eq. (15) can be written as:

$$F(t) = P_0\,\mathrm{Sin}\,t \tag{19}$$

Laplace transform of this load function is given as follows:

$$\bar{F}(s) = P_0\frac{\pi^2(1+e^{-\pi s})}{\pi^2 s^2 + \pi^2} = P_0\frac{1+e^{-\pi s}}{1+s^2} \tag{20}$$

The non-dimensional damping ratio is ξ=0.01 for this problem. The values of ω*=1 rad/s, m=1 kg and P_0=1 kg-cm/s^2 have been chosen. First, discrete values of Eq. (18) are generated for a sequence of Laplace transform parameter and the Modified Durbin's algorithm based on the Fast Fourier Transform (FFT) sub-program is used for the inversion of the results back to time domain. For this purpose computer programs are coded to analyze the dynamic solutions of the problem in the Laplace domain. The accuracy of the numerical solutions obtained has been shown by comparing with the exact solutions of the problem. Numerical results obtained with various Laplace parameters (N =64, 128, 256) with time increments (dt = 0.64 s., 0.32 s., 0.16 s.) and analytical solution of the problem have been shown in the Fig. 7.

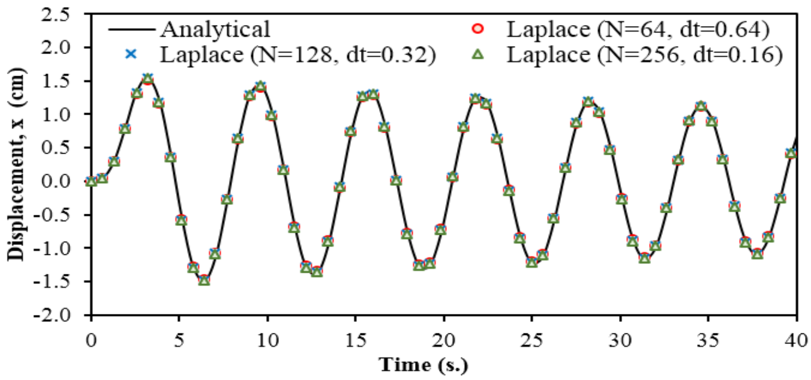

Fig. 7. System displacement versus time (with time interval T=40.96 s.)

Figure 7. shows that the solutions obtained from considering different Laplace parameters or time intervals also overlap with the analytical solution.

IV. CONCLUSION

In the presented method, the governing equations of motion are first obtained in the time domain. Subsequently, Laplace transform is applied and the transformed time-independent problem is solved numerically.The solutions obtained in the Laplace domain are transformed to the time domain with the modified Durbin's numerical inverse Laplace transform method efficiently. On the basis of the present model the following conclusions can be deduced:

In this study, analytical Laplace and numerical Laplace transform methods based on FFT are compared. Since Analytical Laplace transformation requires a lot of computing time, it has been shown that the numerical direct Laplace transformation method based on FFT can be used more effectively.

The result obtained from Durbin's and Durbin's modified inverse Laplace transform methods are compared with the analytical solutions. Durbin's modified inverse Laplace transform methods give results that overlap with analytical results.

The solutions of a mechanical problem using Laplace transform are also compared with their analytical solutions. The solutions obtained in the Laplace domain using even very few parameters overlap with the analytical solutions.

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