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Matrix Geometric Method for M/M/1 Queueing Model With And Without Breakdown ATM Machines

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ABSTRACT: The matrix geometric method is used to derive the stationary distribution of the M/M/1 queueing model with breakdown using the transition structure of its Markov Chain. The M/M/1 queueing model for the ATM machine using the matrix geometric method will be introduced. We study the stability of the two systems with and without breakdown to obtain expressions of the performance measures for the two models. Finally, a comparative study was introduced between the M/M/1 queueing models with and without breakdown.

Keyword: Queues, ATM, Breakdown, Matrix geometric method, Markov Chain.

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I. INTRODUCTION

Queueing theory and related models are used to approximate real queueing situations, so the queueing behavior can be analyzed mathematically; one of these queueing situations is the automated teller machine (ATM) which is one of the several electronic banking channels. It is among the most important service facilities in the banking industry. In classical queueing models, it is common to assume that the server has not failed or breakdown. However, in real life situations, a queueing system might suddenly breakdown and hence the server will not be able to continue providing service unless the system is repaired.The ATM system is one of the systems which always needs some maintenance, it needs to repair if it breakdowns and it needs to recharge with cash if it runs out of money and this case called also breakdown.Server breakdown is an avoidable phenomenon in the service facility of queueing systems. Single server queueing models with breakdown have many applications, for instance, in manufacturing systems, computer systems, communication systems, networks, production systems and many others of systems which might suddenly breakdown.In recent years, queueing models with server failure or breakdown have emerged as one of the important areas of queueing theory. Avi-Itzhak and Naor [2] presented five interesting models of queueing problems with service station subject to breakdown. Numerous researchers including Federgruen and So [4], Takine and Sengupta [11], Núñez-Queija [8] were interested in the breakdown models. Wang *et al*. [12] elaborated on an interesting approach to estimate the equilibrium distribution for the number of customers in the $M^{[x]}/M/1$ queueing model with multiple vacations and server breakdowns. Kumar *et al*. [5] used the probability generating functions to derive the stationary distribution of M/M/1 queueing model with multiple vacation and server breakdown. Raj and Chandrasekar [9] used the matrix geometric method and quasi-birth death process for analyzing the N-policy multiple vacation queueing models with server breakdown and repair; they also provided a numerical example on their model. Ashour *et al*. [1] derived the M/M/1 queueing model with multiple vacations using the matrix geometric method and the transition structure of its Markov Chain. Some performance measures were calculated and a numerical illustration was given to compare the cases of M/M/1 with and without vacations. The Matrix Geometric Method is a useful tool for solving the more complex queueing problems. It is applied by many researchers to solve various queueing problems in different frame works. Neuts [7] explained various matrix geometric solutions of stochastic models. The matrix geometric method is also utilized to develop the computable explicit formula for the probability distributions of the queue length and other system measures of performance. This paper is organized as follows. In Section II, we introduce some of the queueing systems notations, In Section III, we introduce the M/M/1 queueing model for the ATM machine using the matrix geometric method and calculate some measures of performance with given numerical illustration. Section IV. devoted to derive the M/M/1 queueing model with server subject to breakdown by using the same method.

Some measures of performance are calculated and a numerical example is introduced. A comparative study between the two models with and without breakdown will be done in Section V. Finally, the conclusion was introduced in Section VI.

II. SOME QUEUEING SYSTEM NOTATIONS

The M/M/1 queueing model will be derived using the matrix- geometric method and there are some notations will be used in this paper as following:

1. λ **:** The arrival rate,
2. μ : The service rate,
3. η : The breakdown rate,
4. θ : The repair rate,
5. n : The number of customers in the system,
6. ρ : The traffic intensity for the system,
7. π_0 :The probability of idle system,
8. U : The state space,
9. Q : Infinite generate matrix,
10. R : The rate matrix.

III. THE M/M/1 QUEUEING MODEL WITH MATRIX GEOMETRIC METHOD

The M/M/1 queueing model will be established and there are some assumptions need to be achieved in order to keep the model stable which was summarized as following:

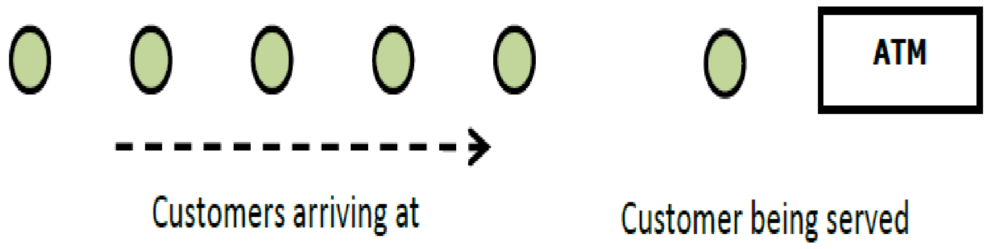

Figure 1: The ATM queueing model**.**

1. Considering one server (one ATM machine) as shown in Figure 1.
2. Customers arrive independently at the ATM machine according to Poisson distribution with rate λ .
3. The service time is exponentially distributed with rate μ .
4. First-come-first-served (FCFS) is the queue discipline for the system.
5. Infinite queue length for the system.
6. The stability condition of the system is $\lambda < \mu$. (See Bakari *et al*.**[3]**).

3.1 The transition rates of the model

In this subsection, the queueing model is formulated by a Quasi Birth and Death (QBD) Markov chain. The Markov chain and transition structure were given by Neuts [7] where,

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ & A_2 & A_1 & A_0 & 0 & \cdots \\ & & A_2 & A_1 & A_0 & \cdots \\ & & \ddots & \ddots & \ddots & \end{bmatrix},$$

which is tri-diagonal matrix, where $B_0 = [-\lambda]$, $A_0 = [\lambda]$, $A_1 = -[\lambda + \mu]$ $and A_2 = [\mu]$.

3.2 Stability condition

Now, we derive the condition for the system to reach the steady state, where the standard drift condition (See Latouche and Ramaswami [6]) which is necessary and sufficient condition for the stability and the QBD Markov chain which is given by:

$$A_0 < A_2 \tag{1}$$

where the stability condition is satisfied as; $\lambda < \mu$ ($A_0 < A_2 \Rightarrow \lambda < \mu$).

3.3 Stationary distribution of the model

From Neuts [7] we get from the stationary distribution of the model,

$$\pi_0 = 1 - \rho \tag{2}$$

and

$$\pi_n = (1-\rho)\rho^n \tag{3}$$

3.4The measures of performance

Firstly, we start by deriving the expected number of customer in the system (L_s) as follows:

$$L_s = E(n) = \sum_{n=0}^{\infty} n\pi_n \tag{4}$$

Thus,

$$L_s = \frac{\rho}{(1-\rho)} \tag{5}$$

Secondly, using Little's law to get:

1. The expected waiting time of customer in the system: $W_s = \frac{L_s}{\lambda}$
2. The expected number of customer in the queue: $L_q = L_s - (1-\pi_0)$
3. The expected waiting time of customer in the queue: $W_q = \frac{L_q}{\lambda}$.

3.5 Numerical illustration for the M/M/1 queueing model

A numerical illustration will be given for the ATM machine using the M/M/1 queueing model.We recorded the arrival and the departure times for 80 customers from one of the ATM machines during two rush hours. Used the chi square goodness of fit test to a specific distributions (Poisson for arrivals and exponential for service times) consequently we obtained that, the arrival rate is λ =40.2 customer/hour and the service rate is μ =55.2 customer/hour for the ATM machine.

To calculate the measures of performance for the M/M/1 queueing model with the values of arrival rate and service rate as following;

- $\rho = \frac{\lambda}{\mu}$ =0.7283=73% ,
- $\pi_0 = 1-\rho$ =0.27=27%,
- $L_s = \frac{\rho}{(1-\rho)}$ =2.7 customers,
- $W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$ =0.07 hour,
- $L_q = L_s - (1-\pi_0)$ =1.95 customers,
- $W_q = \frac{L_q}{\lambda}$ =0.049 hour.

The traffic intensity (ρ) is directly proportional with the arrival rate λ, where it is 73%. The expected number of customers in the system is 2.7 customers, the queue length is 1.95, the expected waiting time of customers in the system is 0.07 hour and the expected waiting time of customers in the queue is 0.049 hour.

IV. M/M/1 QUEUEING MODEL WITH BREAKDOWN

The M/M/1: FCFS queueing model with breakdown will be derived for the ATM machine under same assumptions from Section 2. with the addition of the following assumptions:

1- The breakdown time is exponentially distributed with rate η.

2-The repair time is exponentially distributed with rate θ.

3-Idle and Busy periods occur alternatively so, any time t belongs to either one of these periods where,

- Idle period: a period at which server is breakdown.
- Busy period: a period at which the server is busy (active).

4.1 Markov Chain And Transition Rates Of The Model

In this subsection the queueing model is also formulated by a QBD Markov chain. The Markov chain and transition structure were given with assuming that t ≥ 0 we define:

- $X(t)$: The number of customers in the system.
- $Y(t)$: The states of the server.

Then, consider the two-dimensional stochastic process $\{(X(t),Y(t));t\geq 0\}$, where,

$$Y(t)=\begin{cases}0 & \text{if the server is breakdown.}\\ 1 & \text{if the server is busy.}\end{cases}$$

where, $X(t)$ is level process and $Y(t)$ is the background process.

Obviously, the process $\{(X(t),Y(t));t\geq 0\}$ is a continuous-time Markov chain with the following partitioned state space:

$$U=\bigcup_{l=0}^{\infty}U_l\ , \tag{5}$$

where

$$U_0=\{0\}\times\{0\}=\{(0,0)\} \tag{6}$$

and

$$U_r=\{r\}\times\{0,1\}=\{(r,0),(r,1)\},r\geq 1 \tag{7}$$

where, the elements of the sets are arranged in lexicographical order. (See Ashour *et al*. [1]).

Then the infinitesimal generator matrix is given by:

$$Q=\begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & \cdots \\ & A_2 & A_1 & A_0 & 0 & \cdots \\ & & A_2 & A_1 & A_0 & \cdots \\ & & \ddots & \ddots & \ddots & \end{bmatrix},$$

which is tri-diagonal matrix. Hence $\{(X(t),Y(t));t\geq o\}$ is of the type (QBD) Markov chain.

$$B_{00}=[-\lambda],\ B_{01}=[\lambda\quad 0],\ B_{10}=\begin{bmatrix}0\\ \mu\end{bmatrix}$$

$$A_0=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix},\ A_1=\begin{bmatrix}-(\lambda+\theta) & \theta\\ \eta & -(\lambda+\mu+\eta)\end{bmatrix} \text{ and } A_2=\begin{bmatrix}0 & 0\\ 0 & \mu\end{bmatrix}$$

4.2 Stability condition

To derive the condition for the system to reach the steady state, define the matrix $A=A_0+A_1+A_2$ so:

$$A = \begin{bmatrix} -\theta & \theta \\ \eta & -\eta \end{bmatrix}$$

Clearly, The Continuous Time Markov Chain with generator A is reducible with absorbing state $(1,1)$ and with stationary vector $x = (0,1)$. Then the standard drift condition (See Latouche and Ramaswami [6]) which is necessary and sufficient condition for the stability and the QBD Markov chain is given by:

$$xA_0e < xA_2e \Rightarrow \lambda < \mu \tag{8}$$

where, e is a column vector with all elements equal to ones.

4.3 Stationary distribution of the model

To derive the stationary distribution of the model, we consider the balance equation:

$$\pi Q = 0 \tag{9}$$

where:

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots), \ \ \pi_0 = \pi_{00} and \ \ \pi_n = (\pi_{n0}, \pi_{n1}), n \geq 1 \tag{10}$$

This leads to the following system of equations:

$$\pi_0 B_{00} + \pi_1 B_{10} = 0 \tag{11}$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_2 = 0 \tag{12}$$

$$\pi_1 A_0 + \pi_2 A_1 + \pi_3 A_2 = 0 \tag{13}$$

Given the geometric relation:

$$\pi_n = \pi_1 R^{n-1}, n \geq 2 \tag{14}$$

For stability purposes the spectral radius of R must be less than one.Using Equation (14) to substitute in Equation (13) and divided by π_1 then get:

$$A_0 + RA_1 + R^2 A_2 = 0 \tag{15}$$

where the matrices A_2, A_1, A_0 are given from Subsection 3.1, and by solving Equation (15) using software the rate matrix R will be in the form:

$$R = \frac{\lambda}{\mu} \begin{bmatrix} \frac{(\eta+\mu)}{(\lambda+\theta)} & 1 \\ \frac{\eta}{(\lambda+\theta)} & 1 \end{bmatrix}$$

The normalization condition is given by:

$$\sum_{n=0}^{\infty} \pi_n e = 1 \tag{16}$$

This gives:

$$\pi_0 e + \sum_{n=1}^{\infty} \pi_n e = 1 \tag{17}$$

Using Equation (14) this leads to:

$$\pi_0 e + \pi_1 (I - R)^{-1} e = 1 \tag{18}$$

Since Equation (12) can be written as:

$$\pi_0 B_{01} + \pi_1 (A_1 + RA_2) = 0 \tag{19}$$

Then we will use Equations (11), (18), (19) to find π_0 and π_1, and by using Equation (14) the stationary vector π found.

4.4 The measures of performance

Firstly, we start by deriving the expected number of customer in the system (L_s) as follows:

$$L_s = E(n) = \sum_{n=0}^{\infty} n\pi_n e \tag{20}$$

$$L_s = \pi_1 e + 2\pi_2 e + 3\pi_3 e + \cdots$$

Using Equation (16) to get:

$$L_s = \pi_1 e + 2\pi_1 Re + 3\pi_1 R^2 e + 4\pi_1 R^3 e + \cdots$$

This gives;

$$L_s = \pi_1 (I - R)^{-2} e \tag{21}$$

Secondly, using Little's law to get:

- $W_s = \dfrac{L_s}{\lambda}$,
- $L_q = L_s - (1 - \pi_0)$,
- $W_q = \dfrac{L_q}{\lambda}$.

4.5 Numerical application of the breakdown model

A numerical application will be provided to study the performance of the M/M/1 breakdown model which applied for ATM machine. we use the same data from Section 2. with the same parameters of λ and μ and by assuming values of the parameters η and θ in order to calculate the measures of performance for the same ATM machine but with case of breakdown.

Calculating the measures of performance with the parameters of ($\lambda = 40.2$, $\mu = 55.2$, $\eta = 6$ and $\theta = 54$) as follows:

- $\pi_0 = 0.109$ =0.11
- $L_s = 4.48$ customers,
- $W_s = 0.11$ hour,
- $L_q = 3.58$ customers,
- $W_q = 0.09$ hour.

The expected number of customers in the system is 4.48 customers, the queue length is 3.58, the expected waiting time of customers in the system is 0.11 hour and the expected waiting time of customers in the queue is0.09 hour.

V. A COMPARATIVE STUDY

We will compare between the M/M/1 queueing model with and without breakdown according to the measures of performance, where Table 4. illustrates the results of the measures of performance for the two models.

Table I: Numerical Results For The Models With And Without Breakdown

	Breakdown Model					**Without-Breakdown Model**				
	$\mu = 52, \eta = 6, \theta = 54$					$\mu = 55.2$				
Arrival Rate	L_s	W_s	L_q	W_q	$\pi_0 = P_0$	L_s	W_s	L_q	W_q	$\pi_0 = P_0$
λ =40.2	4.48	0.11	3.58	0.09	0.11	2.7	0.07	1.95	0.049	0.27

By analyzing the whole results, it is observed that from Table 4. the expected number of customers in the system and in the queue for breakdown model is greater than the expected number of customers in the system and in the queue for the model without breakdown. The expected waiting time of customers in the system and in the queue for the breakdown model is greater than the expected waiting time of customers the system and in the queue for the model without breakdown.

VI. CONCLUSION

The matrix geometric method is more convenient to use rather than the classical methods for some queueing models such as M/M/1 with server breakdown.

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