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Some Turing Patterns For The Leslie-Gower Competition Model

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Abstract
In this paper, Turing instability of a double diffusion of Leslie-Gower competition model is considered. Then a series of numerical simulations of the discrete model are performed with diffrent parameters, which get the strip type wave and speckle pattern.

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I. Introduction

Reaction-diffusion systems have been proposed as mechanisms for biological pattern formation in embryological and ecological context, see Murray [10] or Sun et. al. [15]. All such works are based on the pioneering work of [16]. Segel and Jackson [14] were seemingly the first to call attention to the fact that Turing's idea would be applicable in ecological situation also. They conjectured that the nature of the equations which describe chemical interaction does not seem fundamentally different from the nature of those which describe ecological interaction among the species. Again, the idea that dispersal could give rise to instabilities and hence to spatial pattern was due to a number of authors (see [11] or [15], for review) can not cause the Turing's instability.

Then, we have a natural problem. Can the discrete competitive Leslie-Gower system produce Turing instability? Indeed, Turing instabilities of the discrete versions are respectively considered in [2], [6] and [7]. When the diffusion co- efficients are equivalent and the periodic boundary values are added, the wave patterns and the spiral patterns are observed. Furthermore, there are also the different statements for the space- and time-discrete model, the dynamical be- haviors of activator and inhibitor from t to $t + 1$ contains two distinctly different processes, one is the "reaction" stage, the other is "dispersal" stage, for example, see Mistro et al. [13] and [9], Punithan et al. [12], Huang et al. [3] and [4] for the predator-prey model, competitive system ([2] and [6]), diffusion-migeration systems [19], statistical physics [1], Gierer–Meinhardt system [17] and so on.

The present paper is motivated by [7] and [18], the Turing instability or the diffusion-driven instability will be considered for the Leslie-Gower competition model

$$\begin{cases} x_{t+1} = \frac{(1+\varphi_1(h))x_t}{1+\varphi_1(h)(x_t+\varepsilon_2 y_t)}, \\ y_{t+1} = \frac{(1+\varphi_2(h))y_t}{1+\varphi_2(h)(\varepsilon_1 x_t+y_t)}, \end{cases} \tag{1}$$

which can be obtained by using the nonstandard discretization scheme, where

$$\varphi_1(h) = e^h - 1,\ \varphi_2(h) = e^{\gamma h} - 1,$$

and $h > 0$ is the time stepsize, Liu and Elaydi [8]. For convenience, $\varphi_1(h)$ and $\varphi_2(h)$ will be denoted by φ_1 and φ_2, respectively.

Clearly, Systems (1) exists a same positive equilibrium

$$E = (x^*, y^*) = \left(\frac{1-\varepsilon_2}{1-\varepsilon_1\varepsilon_2}, \frac{1-\varepsilon_1}{1-\varepsilon_1\varepsilon_2}\right) \tag{2}$$

when the conditions $0 < \varepsilon_1, \varepsilon_2 < 1$ holds. In view of Theorem 4 in [8], we know that the positive equilibrium E of (1) is globally asymptotically stable in this case. When the diffusion coefficients and the Neumann boundary values are added, the positive equilibrium E becomes unstable and we find lots of Turing patterns.

II. Turing Instability with double diffusion

First of all, we consider the discrete eigenvalues problem of the form

$$\begin{cases} -\Delta u_{ij} = \lambda u_{ij}, (i,j) \in [1,m] \times [1,n], \\ u_{i,0} = u_{i,1}, u_{i,n} = u_{i,n+1}, i \in [0, m+1], \\ u_{0,j} = u_{1,j}, u_{m,j} = u_{m+1,j}, j \in [0, n+1], \end{cases} \tag{3}$$

where

$$\Delta u_{ij} = u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij}.$$

By using the separation variables method, see Zhang, Zhang and Yan [18], we can obtain the eigenvalues

$$\lambda_{kl} = 4\left[\sin^2 \frac{(k-1)\pi}{2m} + \sin^2 \frac{(l-1)\pi}{2n}\right] \tag{4}$$

and the corresponding eigenfunctions

$$\varphi_{ij}^{(kl)} = \cos\frac{(k-1)(2i-1)\pi}{2m} \cos\frac{(l-1)(2j-1)\pi}{2n}, \tag{5}$$

for $(i,j), (k,l) \in [1,m] \times [1,n]$.

In the following, we consider the algebraic equation

$$P(\lambda) \triangleq \lambda^2 + b\lambda + c = 0. \tag{6}$$

which is important for the stability analysis of discrete systems. It is well known that it has two roots of the form

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

By simple calculate, we can obtain the following result:

Proposition 1. The roots $\lambda_{1,2}$ of the algebraic equation (6) satisfy the condition $|\lambda_{1,2}| < 1$ if, and only if $P(1) = 1 + b + c > 0$, $P(-1) = 1 - b + c > 0$ and $P(0) = c < 1$.

Let

$$f(x,y) = \frac{(1+\varphi_1)x}{1+\varphi_1(x+\varepsilon_2 y)} \text{ and } g(x,y) = \frac{(1+\varphi_2)y}{1+\varphi_2(\varepsilon_1 x + y)}.$$

Note that

$$\frac{1+\varphi_1}{1+\varphi_1(x^* + \varepsilon_2 y^*)} = \frac{1+\varphi_2}{1+\varphi_2(\varepsilon_1 x^* + y^*)} = 1,$$

thus, we can get that

$$f_x(x^*, y^*) = 1 - \frac{\varphi_1 x^*}{1+\varphi_1},$$

$$f_y(x^*,y^*)=\frac{-\varepsilon_2\varphi_1 x^*}{1+\varphi_1},$$

$$g_x(x^*,y^*)=\frac{-\varepsilon_1\varphi_2 y^*}{1+\varphi_2},$$

$$g_y(x^*,y^*)=1-\frac{\varphi_2 y^*}{1+\varphi_2}.$$

Then, the Jacobian matrix associated with the linearized system of (1) is

$$J=\begin{pmatrix} \frac{1+\varepsilon_2\varphi_1 y^*}{1+\varphi_1} & \frac{-\varepsilon_2\varphi_1 x^*}{1+\varphi_1} \\ \frac{-\varepsilon_1\varphi_2 y^*}{1+\varphi_2} & \frac{1+\varepsilon_1\varphi_2 x^*}{1+\varphi_2} \end{pmatrix}.$$

Correspondingly, the characteristic equation is

$$\begin{aligned} P(\lambda) &= \begin{vmatrix} 1-\lambda-\frac{\varphi_1 x^*}{1+\varphi_1} & -\frac{\varepsilon_2\varphi_1 x^*}{1+\varphi_1} \\ -\frac{\varepsilon_1\varphi_2 y^*}{1+\varphi_2} & 1-\lambda-\frac{\varphi_2 y^*}{1+\varphi_2} \end{vmatrix} \\ &= \left(1-\lambda-\frac{\varphi_1 x^*}{1+\varphi_1}\right)\left(1-\lambda-\frac{\varphi_2 y^*}{1+\varphi_2}\right) \\ &\quad -\frac{\varepsilon_1\varepsilon_2\varphi_1\varphi_2 x^* y^*}{(1+\varphi_1)(1+\varphi_2)}. \end{aligned}$$

Thus, we immediately get that

$$P(1)=\frac{(1-\varepsilon_1\varepsilon_2)\varphi_1\varphi_2 x^* y^*}{(1+\varphi_1)(1+\varphi_2)},$$

$$P(-1)=4-2\left(\frac{\varphi_1 x^*}{1+\varphi_1}+\frac{\varphi_2 y^*}{1+\varphi_2}\right)+P(1),$$

and that

$$P(0)=1-\left(\frac{\varphi_1 x^*}{1+\varphi_1}+\frac{\varphi_2 y^*}{1+\varphi_2}\right)+P(1).$$

Note that $0<\varepsilon_1,\varepsilon_2<1$ and

$$\begin{cases} x^*+\varepsilon_2 y^*=1, \\ \varepsilon_1 x^*+y^*=1, \end{cases}$$

then, we can know that

$$0<\frac{\varphi_1 x^*}{1+\varphi_1}<1 \text{ and } 0<\frac{\varphi_2 y^*}{1+\varphi_2}<1.$$

In this case, we can obtain that

$$0<P(1)<1,\ P(-1)>0, \text{ and } P(0)<1.$$

which means that the equilibrium point (x^*,y^*) of the system (1) is globally asymptotic stability.

Now, we consider the reaction-diffusion system of the form

$$\begin{cases} u_{ij}^{t+1}=d_1\nabla^2 u_{ij}^t+\frac{(1+\varphi_1)u_{ij}^t}{1+\varphi_1\left(u_{ij}^t+\varepsilon_2 v_{ij}^t\right)}, \\ v_{ij}^{t+1}=d_2\nabla^2 v_{ij}^t+\frac{(1+\varphi_2)v_{ij}^t}{1+\varphi_2\left(\varepsilon_1 u_{ij}^t+v_{ij}^t\right)}, \end{cases} \tag{7}$$

with the Neumann boundary conditions

$$\begin{cases} u_{i,0}^t=u_{i,1}^t, u_{i,n}^t=u_{i,n+1}^t, v_{i,0}^t=v_{i,1}^t, v_{i,n}^t=v_{i,n+1}^t, i\in[0,m+1], \\ u_{0,j}^t=u_{1,j}^t, u_{m,j}^t=u_{m+1,j}^t, v_{0,j}^t=v_{1,j}^t, v_{m,j}^t=v_{m+1,j}^t, j\in[0,n+1], \end{cases} \tag{8}$$

for $(i,j)\in[1,m]\times[1,n]$ and $t\in Z^+$, where m and n are positive integers, $d_1,d_2>0$ are the diffusion coefficients,

$$\nabla^2 u_{ij}^t = u_{i+1,j}^t + u_{i,j+1}^t + u_{i-1,j}^t + u_{i,j-1}^t - 4u_{ij}^t$$

and

$$\nabla^2 v_{ij}^t = v_{i+1,j}^t + v_{i,j+1}^t + v_{i-1,j}^t + v_{i,j-1}^t - 4v_{ij}^t.$$

Clearly, (x^*, y^*) is also positive equilibrium of (7)-(8), thus, the linearized form of (7) in $(x^*, y^*) = (u^*, v^*)$ is

$$\begin{cases} u_{ij}^{t+1} = d_1 \nabla^2 u_{ij}^t + f_u u_{ij}^t + f_v v_{ij}^t, \\ v_{ij}^{t+1} = d_2 \nabla^2 v_{ij}^t + g_u u_{ij}^t + g_v v_{ij}^t, \end{cases} \tag{9}$$

where $f_u = f_x(x^*, y^*)$, $f_v = f_y(x^*, y^*)$, $g_u = g_x(x^*, y^*)$, and $g_v = g_y(x^*, y^*)$.

Then respectively taking the inner product of (9) with the corresponding eigenfunction $\varphi_{ij}^{(kl)}$ of the eigenvalue λ_{kl} for $(i, j), (k, l) \in [1, m] \times [1, n]$ and use the Neumann boundary conditions (8), we see that

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^{t+1} &= d_1 \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} \nabla^2 u_{ij}^t + f_u \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t \\ &\quad + f_v \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t \\ &= d_1 \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}^t \nabla^2 \varphi_{ij}^{(kl)} + f_u \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t \\ &\quad + f_v \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t \\ &= -d_1 \lambda_{kl} \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}^t \varphi_{ij}^{(kl)} + f_u \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t \\ &\quad + f_v \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^{t+1} &= d_2 \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} \nabla^2 v_{ij}^t + g_u \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t \\ &\quad + g_v \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t \\ &= -d_2 \lambda_{kl} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t + g_u \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t \\ &\quad + g_v \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t. \end{aligned}$$

Let $U^t = \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} u_{ij}^t$ and $V^t = \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi_{ij}^{(kl)} v_{ij}^t$, then we have

$$\begin{cases} U^{t+1} = f_u U^t + f_v V^t - d_1 \lambda_{kl} U^t \\ V^{t+1} = g_u U^t + g_v V^t - d_2 \lambda_{kl} V^t \end{cases}$$

or

$$\begin{cases} U^{t+1} = (f_u - d_1 \lambda_{kl}) U^t + f_v V^t \\ V^{t+1} = g_u U^t + (g_v - d_2 \lambda_{kl}) V^t. \end{cases} \tag{10}$$

If (U^t, V^t) is a solution of the system (10), then $\left(u_{ij}^t = U^t \varphi_{ij}^{(kl)}, v_{ij}^t = V^t \varphi_{ij}^{(kl)}\right)$ is also clearly a solution of (9) with the Neumann boundary conditions (8). Thus, the unstable system (10) will produce that the problem (9)-(8) is untable.

Proposition 2. If there exists some eigenvalue λ_{kl} of (3) such that system (10) is untable, then the positive equilibrium (x^*, y^*) of (7)-(8) is also untable.

The Jacobian matrix of system (10) is

$$J^* = \begin{pmatrix} \frac{1+\varepsilon_2\varphi_1 y^*}{1+\varphi_1} & \frac{-\varepsilon_2\varphi_1 x^*}{1+\varphi_1} \\ \frac{-\varepsilon_1\varphi_2 y^*}{1+\varphi_2} & \frac{1+\varepsilon_1\varphi_2 x^*}{1+\varphi_2} \end{pmatrix} - \begin{pmatrix} d_1\lambda_{kl} & 0 \\ 0 & d_2\lambda_{kl} \end{pmatrix},$$

correspondingly, the characteristic function is

$$\begin{aligned} \overline{P}(\lambda) &= \left(1 - d_1\lambda_{kl} - \lambda - \frac{\varphi_1 x^*}{1+\varphi_1}\right)\left(1 - d_2\lambda_{kl} - \lambda - \frac{\varphi_2 y^*}{1+\varphi_2}\right) \\ &\quad - \frac{\varepsilon_1\varepsilon_2\varphi_1\varphi_2 x^* y^*}{(1+\varphi_1)(1+\varphi_2)} \end{aligned}$$

which implies that

$$\overline{P}(1) = \lambda_{kl}\left(d_1 d_2\lambda_{kl} + \frac{d_1\varphi_2 y^*}{1+\varphi_2} + \frac{d_2\varphi_1 x^*}{1+\varphi_1}\right) + P(1),$$

$$\overline{P}(-1) = \lambda_{kl}\left[d_1 d_2\lambda_{kl} - d_2\left(2 - \frac{\varphi_1 x^*}{1+\varphi_1}\right) - d_1\left(2 - \frac{\varphi_2 y^*}{1+\varphi_2}\right)\right] + P(-1),$$

and

$$\overline{P}(0) = \lambda_{kl}\left[d_1 d_2\lambda_{kl} - d_2\left(1 - \frac{\varphi_1 x^*}{1+\varphi_1}\right) - d_1\left(1 - \frac{\varphi_2 y^*}{1+\varphi_2}\right)\right] + P(0).$$

which can cause Turing pattern. Calculating these inequalities is difficult, but we can rely on numerical simulations in the next section.

III. Numerical simulation

For simplicity, we let $d_1 = d_2 = d$, $\varphi_1 = \varphi_2 = \varphi$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and we provide three examples that satisfy the conditions (11). In the following, a series of numerical simulations will be performed so that we can explore the dynamical behavior of the discrete competition system (7) with the conditions (8). In all of the following simulations, the small amplitude random perturbation is 0.001 around the steady state, the size of the lattice is chosen to be 200×200, and set $\lambda = \max(\lambda_{kl}) \approx 0.7998$.

Simulations of pattern development at $t = 1000; 2000; 5000; 20000$, which shows the evolution in the spacing as the interaction time proceeds. Here we choose three different sets of parameters, which can show us the speckle pattern (see fig.1) and the strip type (see fig.2).

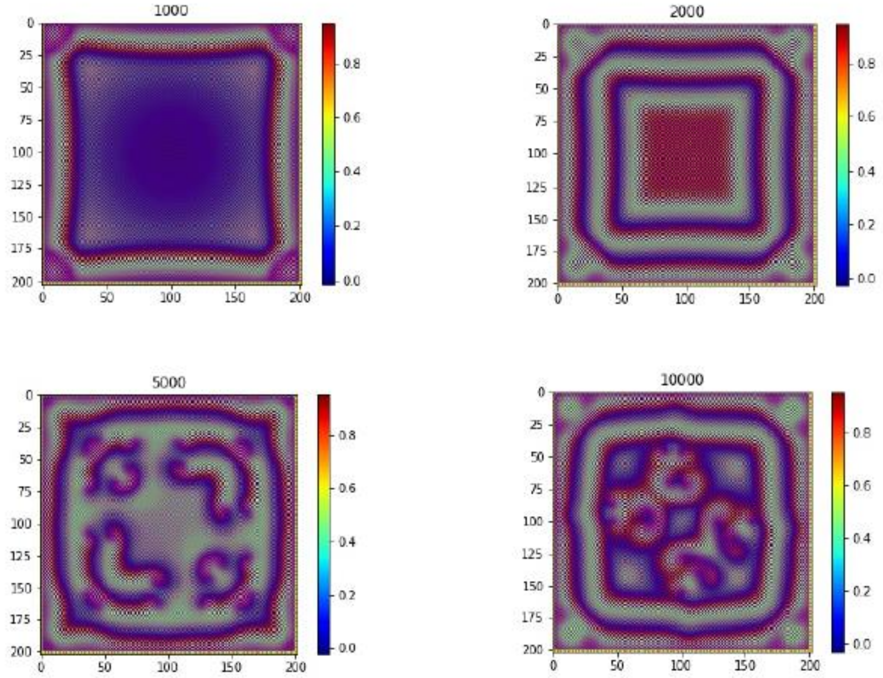

Fig.1. the diagram when $\varphi = 1, \varepsilon = 0.5, d = 0.221$

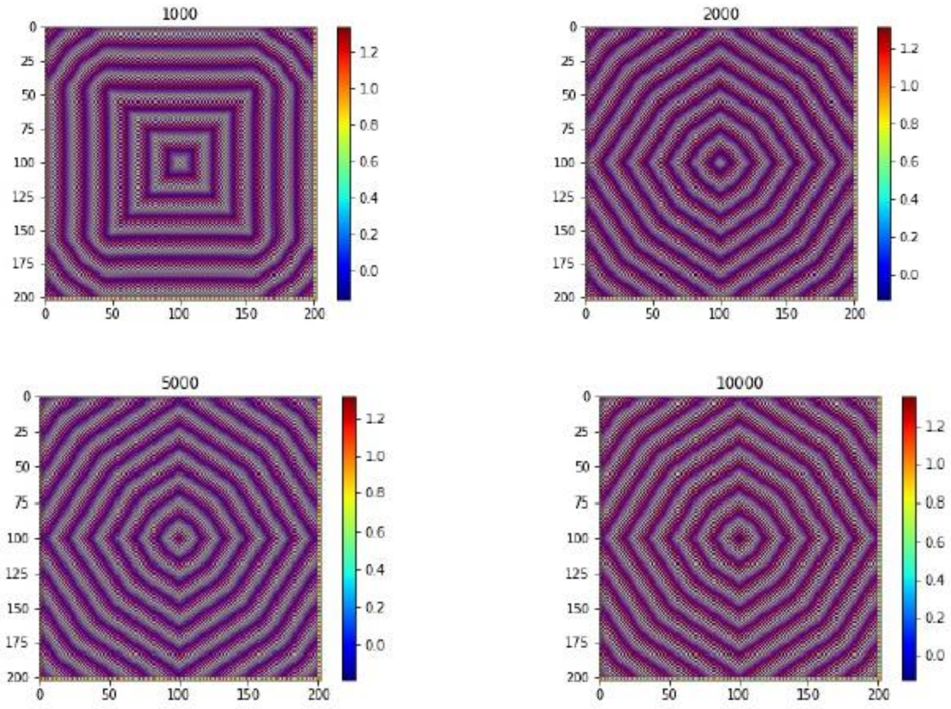

Fig.2. the diagram when $\varphi = 1.4, \varepsilon = 0.5, d = 0.23$

IV. Conclusions

Firstly, we have presented a theoretical analysis of Turing instability for Leslie- Gower competition model, and give some condition. Secondly, a large variety of Turing pattern are obtained by numerical simulations which is consistent with the predictions drawn from the analysis of the discrete competitive system.

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