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Generalization of the Fractional Order Economic OrderQuantity Model with Cubic Demand Rate and Salvage Value

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ABSTRACT :*In this study, generalizations were discussed when the memory effect was added to the EOQ supply model with cubic demand levels and salvage values. In the EOQ model, the influence of memory effects plays a role in determining the optimal ordering interval and the minimized total average cost. The classic EOQ model was converted into a fractional order EOQ model using the Caputo derivative definition. There are two types of memory index, namely the differential memory index* (α)*which corresponds to the inventory rate/level and the integral memory index*(β) *which corresponds to the fractional order integral to determine the storage cost of inventory problems. The primal geometric programming method was introduced in solving fractional order inventory models. Based on numerical simulation results, the influence of differential memory index is more active than integral memory index.*
KEYWORDS Fractional order derivative, Fractional Laplace transform method, salvage value

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I. INTRODUCTION

The inventory model is formulated for the business purpose of determining optimal ordering intervals by minimizing total costs and optimal inventory levels. In the inventory system, the optimal ordering interval and the minimum total cost depend on some variable of the inventory system. These variables depend on the environment of the store or company i.e. the position of the store or company, the political or social situation. In addition, the decrease or increase in profits depends on the relationship of company staff or shopkeepers towards customers, if customers get a bad experience from the company or store then they will not agree to buy products from the company or store resulting in demand will gradually decrease. Based on such reviews a system that relies on past experience, is called the memory effect of the inventory system. Therefore, the inventory system is a system that depends on memory.

Fractional calculus is a branch of mathematical science that studies integrals and derivatives of fractional order. Fractional calculus was first introduced by L'Hopital and Leibniz in 1695. The fractional differential equation model is more appropriate in natural phenomena than in the integer order, because the subsequent state of affairs in the model depends not only on the current state but also on all previous states. As a result, the fractional derivative at a given point contains information about the function at the previous points. In solving fractional differential equations several definitions are needed[1]. The definitions in question are those of Riemann-Liouville and Caputo. Fractional derivatives and integrals have an important physical interpretation of the memory system or memory effects. So it can be applied to systems that have the effect of past experiences such as in economic models, epidemic models and one of them on inventory systems.

The classical supply model integer order is a regular differential equation containing an integer order derivative with respect to time. The differential equation of the integer order actually only describes the instantaneous change of the inventory level, lacking the power to combine the effects of system memory [8]. But fractional order derivatives and fractional integration are physically treated as memory index and memory strengths that depend on fractional order to put into the inventory model.

Fractional calculus is one of the strongest tools for describing many physical phenomena that are overlooked in the classical model order integer. Its descriptive power is to analyze behavior on a scale small enough that its properties change linearly, as well as avoid complexities that arise on a larger scale [1]. Fractional calculus can be used to describe EOQ models with time-dependent linear demand trends for developing more general EOQ models [2]. The physical meaning of the fractional order is the memory index. Memory means that it depends not only on the current state of the system, but also the state of the previous system [6]. Research related to the effect of memory on supply models with linear demand levels and the presence of salvage values [7].The salvage value is given to the inventory system but the main focus is incorporating the memory effect into the inventory mode [9]l. This memory effect is then applied to EOQ models with non-constant demand rates that are linear. In the EOQ model, the influence of the memory effect plays a role in determining the minimum average total cost [8].The conducted research related to the EOQ inventory model with cubic demand levels and the presence of salvage value and shortage [4].In this study, a fractional order EOQ model with a cubic demand rate with residual values will be discussed. The request rate is considered a cubic function of time-dependent storage time and costs. The model is also accompanied by the residual value of a supply system. The application of primal geometry programming was introduced in solving fractional order EOQ models.

II. REVIEW OF FRACTIONAL CALCULUS

One of the most important basic functions of fractional calculus is that the Gamma function is symbolized by$\Gamma(\alpha)$,which is a generalization of$n!$ and satisfy$n$for non integernumbers and complex numbers. The Gamma function$\Gamma(\alpha)$is defined as following from [10]:

$$\Gamma(\alpha) \quad = \quad \int_0^\infty e^{-t} t^{\alpha-1} dt \text{, untuk } \alpha > 0. \qquad (1)$$

The Beta function is defined as

$$\beta(\alpha, n) \quad = \quad \int_0^1 t^{\alpha-1}(1-t)^{n-1}\, dt$$

convergent for$n > 0$.The Beta function for the limit$(0, T)$is defined as

$$\mathrm{B}(\alpha, n) \quad = \quad \frac{1}{T^{\alpha+n-1}} \int_0^T t^{\alpha-1}(T-t)^{n-1} dt.$$

2.1 Riemann-Liouville Integrals

Let $\alpha \in \mathbb{R}^+$ and $L_1[a, b]$ areLebesgueintegrable functions in $[a, b]$ with $a < b < \infty$, the operator I_a^α is defined in$L_1[a, b]$as follows[3]:

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

for$a \leq t \leq b$and$\Gamma(\alpha)$is a Gamma function.

2.2 Caputo fractional order derivative

The definition of Caputo on fractional derivatives can be written as follows

$$\begin{aligned} {}_aD_t^\alpha f(t) \quad &= \quad I_t^{n-\alpha} D_t^n f(t), \quad n = 1,2, \ldots \\ &= \quad \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \end{aligned}$$

for$-1 < \alpha < n$[10].

2.3 Fractional Laplace transform method

The transformation of the Laplace function$f(t)$is denoted by$F(s)$and is defined as follows [11]

$$F(s) \quad = \quad L\big(f(t)\big) \quad = \quad \int_0^\infty e^{-st} f(t) dt,$$

where$s > 0$and$s$is the transform parameter. The Laplace transform of the integer order derivative is defined as follows

$$L\big(f^m(t)\big) \quad = \quad s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^k(0),$$

where$f^m(t)$indicates the integer order derivative of fagainst t. If$F(s)$is a Laplace transformation of$f(t)$, then the Laplace transform of the Caputo fractional derivative operator with order$\alpha > 0$can be written as follows

$$L\{{}^C D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^k(0),$$

where$(m-1) < \alpha \leq m$.

2.4 Classical model

In this study, the following assumptions were given:

1. Developed model for only one type of goods (single item) and no interaction with other goods

2. The lead time between the time of ordering and delivery of goods is zero or negligible.
3. Storage costs depend on the average number of goods stored.
4. The demand rate is a cubic function

$$a + bt + ct^2 + dt^3,\ a > 0, b, c, d \geq 0,$$

5. There is no shortage.

The parameters C_1 is inventory holding cost, C_3 is ordering cost or setup cost, and $I(t)$ is stock level or inventory level. Inventory reaches zero level at the time $t = T$. Value of an inventory also depends on Q i.e. total order quantity as well as demand level. Therefore, the inventory level at any time during the time interval $[0, T]$ can be represented by a first-order ordinary differential equation as follows

$$\frac{dI(t)}{dt} = -(a + bt + ct^2 + dt^3) \quad (2)$$

2.5 Fractional order inventory model formulation with memory kernel

The fractional order inventory model differents from the classical model in that it considers the degree of fractional change at the inventory level. Then to study the influence of the memory effect, first a differential equation is written using the memory kernel function in the following form[5]

$$\frac{dI(t)}{dt} = -\int k(t - t^{'})(a + bt^{'} + c(t^{'})^2 + d(t^{'})^3)dt^{'} \quad (3)$$

wherememory kernel$k(t - t^{'})$plays an important role in reducing the integer order system to a fractional order system. In reality, any arbitrary function i.e. a function for which there is no direct or mandatory relationship between the symbol and the one it symbolizes can be replaced with a number of delta functions $\delta(t - t^{'})$, thus leading to a certain type of time correlation. So, to produce a fractional order model is assumed$k(t - t^{'}) = \frac{1}{\Gamma(\alpha - 1)}(t - t^{'})^{(\alpha-2)}, 0 < \alpha \leq 1$ and$\Gamma(\alpha)$is Gamma function. Using the definition of a fractional derivative of Caputo, the equation can be written into the form of a fractional differential equation with a Caputo-type derivative in the form as follows,

$$\frac{dI(t)}{dt} = -\,{}_0D_t^{-(\alpha-1)}(a + bt + ct^2 + dt^3).$$

The next step is to apply the fractional derivative of the Caputo order $(\alpha - 1)$ on both sides and using the fact that the fractional derivative of the Caputo and the fractional integral are inverse operators, then the following fractional differential equation model can be obtained:

$${}_0^CD_t^{\alpha} I(t) = -(a + bt + ct^2 + dt^3)$$

Or equivalently

$$\frac{d^{\alpha}\big(I(t)\big)}{dt^{\alpha}} = -(a + bt + ct^2 + dt^3)$$

where$0 < \alpha \leq 1, 0 \leq t \leq T$with boundary conditions$I(T) = 0, I(0) = Qand\alpha$considered a differential memory index. Systems of equations (2) are modified into systems of fractional differential equations. The derivative$\frac{dI}{dt}$is modifiedto$\frac{d^{\alpha}(I(t))}{dt^{\alpha}}$, so that it is obtained

$$\frac{d^{\alpha}\big(I(t)\big)}{dt^{\alpha}} = -(a + bt + ct^2 + dt^3) \quad (4)$$

where$0 < \alpha \leq 1, 0 \leq t \leq T$with boundary conditions$I(T) = 0, I(0) = Qand\alpha$considered a differential memory index. The fractional order inventory model can be solved using the fractional Laplace transformation method with a given initial value. Equation (4) obtained a fractional order supply model against time t, as follows:

$$I(t) = \left(Q - \frac{at^{\alpha}}{\Gamma(1+\alpha)} - \frac{bt^{\alpha+1}}{\Gamma(2+\alpha)} - \frac{2ct^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{6dt^{\alpha+3}}{\Gamma(4+\alpha)}\right) \quad (5)$$

Using the boundary condition $I(0) = Q$ in equation (5) the total number of orders is obtained as follows:

$$Q = \left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{bT^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2cT^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6dT^{\alpha+3}}{\Gamma(4+\alpha)}\right) \quad (6)$$

Furthermore, by substitution of equation (6) into equation (5) so that a model of fractional order supply can be formulated against t time, as follows

$$I(t) = \frac{a(T^\alpha - t^\alpha)}{\Gamma(1+\alpha)} + \frac{b(T^{\alpha+1} - t^{\alpha+1})}{\Gamma(2+\alpha)} + \frac{2c(T^{\alpha+2} - t^{\alpha+2})}{\Gamma(3+\alpha)} + \frac{6d(T^{\alpha+3} - t^{\alpha+3})}{\Gamma(4+\alpha)}. \quad (7)$$

The total purchase/production cost (purchased cost) is the multiplication between the price of the goods (P) and the amount of demand for goods (Q), namely

$$PC_{\alpha,\beta} = P\left(\frac{aT^\alpha}{\Gamma(1+\alpha)} + \frac{bT^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2cT^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6dT^{\alpha+3}}{\Gamma(4+\alpha)}\right) \quad (8)$$

The total cost of β order inventory is denoted as $HOC_{\alpha,\beta}(T)$. The β parameter is an integral memory index relating to transportation costs. Furthermore $HOC_{\alpha,\beta}(T)$ is defined as follows

$$\begin{aligned} HOC_{\alpha,\beta}(T) &= C_1\left({}_0D_T^{-\beta}(I(t))\right) \\ &= \frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{\beta-1} I(t)dt \\ &= \frac{c_1}{\Gamma(\beta)}\left[\frac{a\,T^{\alpha+\beta+1}}{\Gamma(1+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+2,\beta)\right) + \frac{b\,T^{\alpha+\beta+2}}{\Gamma(2+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+3,\beta)\right)\right. \\ &\quad \left. + \frac{2c\,T^{\alpha+\beta+3}}{\Gamma(3+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+4,\beta)\right) + \frac{6d\,T^{\alpha+\beta+4}}{\Gamma(4+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+5,\beta)\right)\right] \end{aligned}$$

Furthermore, the total residual value of the βorder is denoted as $SL_{\alpha,\beta}$ and is defined as follows

$$\begin{aligned} SL_{\alpha,\beta} &= \gamma\left[Q - D^{-\beta}(a + bt + ct^2 + dt^3)\right] \\ &= \gamma\left[\frac{a\,T^\alpha}{\Gamma(1+\alpha)} + \frac{b\,T^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2c\,T^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6d\,T^{\alpha+3}}{\Gamma(4+\alpha)} - \frac{1}{\Gamma(\beta)}\int_0^T (T-t)^{\beta-1}(a + bt + ct^2 + dt^3)dt\right] \\ &= \gamma\left[\frac{a\,T^\alpha}{\Gamma(1+\alpha)} + \frac{b\,T^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2c\,T^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6d\,T^{\alpha+3}}{\Gamma(4+\alpha)}\right. \\ &\quad \left. - \left(\frac{aT^\beta}{\Gamma(\beta+1)} + \frac{b\mathrm{B}(2,\beta)T^{\beta+1}}{\Gamma(\beta)} + \frac{c\mathrm{B}(3,\beta)T^{\beta+2}}{\Gamma(\beta)} + \frac{d\mathrm{B}(4,\beta)T^{\beta+3}}{\Gamma(\beta)}\right)\right] \end{aligned}$$

$\left(SL_{\alpha,\beta} \neq 0 \text{ for } \alpha \neq \beta < 1\right)$.

The average total cost is obtained, which is

$$\begin{aligned} TOC_{\alpha,\beta}(T) &= \frac{\left(PC_{\alpha,\beta} + HOC_{\alpha,\beta}(T) + C_3 - SL_{\alpha,\beta}\right)}{T} \\ &= A\,T^{\alpha+\beta} + B\,T^{\alpha+\beta+1} + C\,T^{\alpha+\beta+2} + D\,T^{\alpha+\beta+3} + ET^{\alpha-1} + F\,T^\alpha \\ &\quad + G\,T^{\alpha+1} + HT^{\alpha+2} + IT^{\beta-1} + JT^\beta + KT^{\beta+1} + LT^{\beta+2} + MT^{-1} \end{aligned} \quad (9)$$

with, $A = \frac{ac_1}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+2,\beta)\right), B = \frac{bc_1}{\Gamma(\beta)\Gamma(2+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+3,\beta)\right),$

$$C = \frac{2cc_1}{\Gamma(\beta)\Gamma(3+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+4,\beta)\right), D = \frac{6dc_1}{\Gamma(\beta)\Gamma(4+\alpha)}\left(\mathrm{B}(2,\beta) - \mathrm{B}(\alpha+5,\beta)\right),$$

$$E = \frac{a(P-\gamma)}{\Gamma(1+\alpha)}, F = \frac{b(P-\gamma)}{\Gamma(2+\alpha)}, G = \frac{2c(P-\gamma)}{\Gamma(3+\alpha)}, H = \frac{6d(P-\gamma)}{\Gamma(4+\alpha)}, I = \frac{\gamma a}{\Gamma(\beta+1)},$$

$$J = \frac{\gamma b\mathrm{B}(2,\beta)}{\Gamma(\beta)}, K = \frac{\gamma c\mathrm{B}(3,\beta)}{\Gamma(\beta)}, L = \frac{\gamma d\mathrm{B}(4,\beta)}{\Gamma(\beta)}, M = C_3$$

III. Primal-Geometric Programming Method

Next will consider the following cases to study the behavior of this fractional order inventory model:

(i)**Case 1: 0** $< \alpha < 1$ **dan 0** $< \beta < 1$

The total average cost in equation (9) i.e.

$$\begin{aligned} TOC_{\alpha,\beta}(T) &= A\,T^{\alpha+\beta} + B\,T^{\alpha+\beta+1} + C\,T^{\alpha+\beta+2} + D\,T^{\alpha+\beta+3} + ET^{\alpha-1} + F\,T^\alpha + G\,T^{\alpha+1} \\ &\quad + HT^{\alpha+2} + IT^{\beta-1} + JT^\beta + KT^{\beta+1} + LT^{\beta+2} + MT^{-1} \end{aligned}$$

$$A=\frac{ac_1}{\Gamma(\beta)\Gamma(1+\alpha)}\left(\mathrm{B}(2,\beta)-\mathrm{B}(\alpha+2,\beta)\right), B=\frac{bc_1}{\Gamma(\beta)\Gamma(2+\alpha)}\left(\mathrm{B}(2,\beta)-\mathrm{B}(\alpha+3,\beta)\right),$$

$$C=\frac{2cc_1}{\Gamma(\beta)\Gamma(3+\alpha)}\left(\mathrm{B}(2,\beta)-\mathrm{B}(\alpha+4,\beta)\right), D=\frac{6dc_1}{\Gamma(\beta)\Gamma(4+\alpha)}\left(\mathrm{B}(2,\beta)-\mathrm{B}(\alpha+5,\beta)\right),$$

$$E=\frac{a(P-\gamma)}{\Gamma(1+\alpha)}, F=\frac{b(P-\gamma)}{\Gamma(2+\alpha)}, G=\frac{2c(P-\gamma)}{\Gamma(3+\alpha)}, H=\frac{6d(P-\gamma)}{\Gamma(4+\alpha)}, I=\frac{\gamma a}{\Gamma(\beta+1)},$$

$$J=\frac{\gamma b\mathrm{B}(2,\beta)}{\Gamma(\beta)}, K=\frac{\gamma c\mathrm{B}(3,\beta)}{\Gamma(\beta)}, L=\frac{\gamma d\mathrm{B}(4,\beta)}{\Gamma(\beta)}, M=C_3.$$

The inventory model can be written as follows:

$$\begin{cases} Min\ TOC_{\alpha,\beta}(T) = A\,T^{\alpha+\beta}+B\,T^{\alpha+\beta+1}+C\,T^{\alpha+\beta+2}+D\,T^{\alpha+\beta+3}+ET^{\alpha-1}+F\,T^{\alpha} \\ \quad +G\,T^{\alpha+1}+HT^{\alpha+2}+IT^{\beta-1}+JT^{\beta}+KT^{\beta+1}+LT^{\beta+2}+MT^{-1} \\ \quad subject\ to\ T\geq 0 \end{cases} \quad (10)$$

The analytical solution of fractional differential equations is not easy to determine so it is necessary to approach it, for example by numerical methods. In this study, equation (10) is a nonlinear equation in the form of a posinomial, that is, the exponent of a posinomial is any real number but the coefficient must be a positive real number. The Geometric program presents the problem of minimizing posinomial-shaped functions called Primal functions and maximizing multiplication functions called Dual functions. Furthermore, in equation (10) the minimum of the dual function will be searched through the maximum of its dual function symbolized by the variable w, as follows:

$$\begin{aligned} Max\ d(w) = & \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}\left(\frac{G}{w_7}\right)^{w_7} \\ & \left(\frac{H}{w_8}\right)^{w_8}\left(\frac{I}{w_9}\right)^{w_9}\left(\frac{J}{w_{10}}\right)^{w_{10}}\left(\frac{K}{w_{11}}\right)^{w_{11}}\left(\frac{L}{w_{12}}\right)^{w_{12}}\left(\frac{M}{w_{13}}\right)^{w_{13}} \end{aligned} \quad (11)$$

where$w_i\,(i=1,2,3,\dots,13)$. Normalized condition is as

$$w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}+w_{11}+w_{12}+w_{13} = 1. \quad (12)$$

orthogonal condition is as

$$\left.\begin{aligned} &(\alpha+\beta)w_1+(\alpha+\beta+1)w_2+(\alpha+\beta+2)w_3+(\alpha+\beta+3)w_4+(\alpha-1)w_5+\alpha w_6 \\ &+(\alpha+1)w_7+(\alpha+2)w_8+(\beta-1)w_9+\beta w_{10}+(\beta+1)w_{11}+(\beta+2)w_{12}-w_{13} \end{aligned}\right\} = 0. \quad (13)$$

and the primal-dual relations are as follows

$$\left.\begin{aligned} &A\,T^{\alpha+\beta}=w_1 d(w), && B\,T^{\alpha+\beta+1}=w_2 d(w), && C\,T^{\alpha+\beta+2}=w_3 d(w) \\ &D\,T^{\alpha+\beta+3}=w_4 d(w), && ET^{\alpha-1}=w_5 d(w), && F\,T^{\alpha}=w_6 d(w) \\ &G\,T^{\alpha+1}=w_7 d(w), && HT^{\alpha+2}=w_8 d(w), && IT^{\beta-1}=w_9 d(w) \\ &JT^{\beta}=w_{10} d(w), && KT^{\beta+1}=w_{11} d(w), && LT^{\beta+2}=w_{12} d(w) \\ &MT^{-1}=w_{13} d(w) \end{aligned}\right\}. (14)$$

Based on the equation (14) obtained the followings are given by

$$\left.\begin{aligned} &\frac{AW_2}{BW_1}=\frac{BW_3}{CW_2}, && \frac{AW_2}{BW_1}=\frac{CW_4}{DW_3}, && \left(\frac{AW_2}{BW_1}\right)^{\beta+4}=\frac{EW_4}{DW_3} \\ &\frac{AW_2}{BW_1}=\frac{EW_6}{FW_5}, && \frac{AW_2}{BW_1}=\frac{FW_7}{GW_6}, && \frac{AW_2}{BW_1}=\frac{GW_8}{HW_7} \\ &\left(\frac{AW_2}{BW_1}\right)^{\alpha-\beta+3}=\frac{IW_8}{HW_9}, && \frac{AW_2}{BW_1}=\frac{IW_{10}}{JW_9}, && \frac{AW_2}{BW_1}=\frac{JW_{11}}{KW_{10}} \\ &\frac{AW_2}{BW_1}=\frac{KW_{12}}{LW_{11}}, && \left(\frac{AW_2}{BW_1}\right)^{\beta+3}=\frac{MW_{12}}{LW_{13}}, && \end{aligned}\right\} (15)$$

along with

$$T^* = \frac{AW_2}{BW_1}. \quad (16)$$

Solving $(12), (13), (15)$ the critical value $w_i^*, i = 1,2, \ldots, 13$ of the dual variable $w_i, i = 1,2, \ldots, 13$ and finally the optimum value T^* of T has been calculated from the equation of (16) substituting the critical values of the dual variable. Now the minimized total average cost $TOC_{\alpha,\beta}(T)$ has been calculated by substituting T^* in (10) analytically. The minimized total average cost and optimal ordering interval is evaluated from (16) numerically using matlab minimization method

(ii) Case 2: $0 < \alpha < 1$ and $\beta = 1$

Therefore, the minimized total average cost becomes as

$$\begin{aligned} TOC_{\alpha,1}(T) &= A\,T^{\alpha+1} + B\,T^{\alpha+2} + C\,T^{\alpha+3} + D\,T^{\alpha+4} + ET^{\alpha-1} + F\,T^{\alpha} \\ &\quad + GT^0 + HT + IT^2 + JT^3 + KT^{-1} \end{aligned} \quad (17)$$

with

$$A = \frac{ac_1}{\Gamma(1)\Gamma(1+\alpha)}\big(\mathrm{B}(2,1) - \mathrm{B}(\alpha+2,1)\big) + \frac{2c(P-\gamma)}{\Gamma(3+\alpha)},$$

$$B = \frac{bc_1}{\Gamma(1)\Gamma(2+\alpha)}\big(\mathrm{B}(2,1) - \mathrm{B}(\alpha+3,1)\big) + \frac{6d(P-\gamma)}{\Gamma(4+\alpha)},$$

$$C = \frac{2cc_1}{\Gamma(1)\Gamma(3+\alpha)}\big(\mathrm{B}(2,1) - \mathrm{B}(\alpha+4,1)\big), D = \frac{6dc_1}{\Gamma(1)\Gamma(4+\alpha)}\big(\mathrm{B}(2,1) - \mathrm{B}(\alpha+5,1)\big),$$

$$E = \frac{a(P-\gamma)}{\Gamma(1+\alpha)}, F = \frac{b(P-\gamma)}{\Gamma(2+\alpha)}, G = \frac{\gamma a}{\Gamma(2)}, H = \frac{\gamma b\mathrm{B}(2,1)}{\Gamma(1)}, I = \frac{\gamma c\mathrm{B}(3,1)}{\Gamma(1)},$$

$$J = \frac{\gamma d\mathrm{B}(4,1)}{\Gamma(1)}, K = C_3.$$

The inventory model can be written as follows:

$$\begin{cases} Min\;TOC_{\alpha,1}(T) = A\,T^{\alpha+1} + B\,T^{\alpha+2} + C\,T^{\alpha+3} + D\,T^{\alpha+4} + ET^{\alpha-1} + F\,T^{\alpha} \\ \quad + GT^0 + HT + IT^2 + JT^3 + KT^{-1} \\ \quad untuk\; T \geq 0. \end{cases} \quad (18)$$

In similar case 1 using primal geometric programming approach, the minimized total average cost and the optimal ordering interval is obtained from (18) analytically.

(iii) Case 3: $\alpha = 1$ and $0 < \beta < 1$

Here, the total average cost is

$$\begin{aligned} TOC_{1,\beta}(T) &= A\,T^{1+\beta} + B\,T^{\beta+2} + C\,T^{\beta+3} + D\,T^{\beta+4} + ET^0 + FT + G\,T^2 \\ &\quad + HT^3 + IT^{\beta-1} + JT^{\beta} + KT^{-1} \end{aligned} \quad (19)$$

with

$$A = \frac{ac_1}{\Gamma(\beta)\Gamma(2)}\big(\mathrm{B}(2,\beta) - \mathrm{B}(3,\beta)\big) + \frac{\gamma c\mathrm{B}(3,\beta)}{\Gamma(\beta)}, \qquad B = \frac{bc_1}{\Gamma(\beta)\Gamma(3)}\big(\mathrm{B}(2,\beta) - \mathrm{B}(4,\beta)\big) + \frac{\gamma d\mathrm{B}(4,\beta)}{\Gamma(\beta)},$$

$$C = \frac{2cc_1}{\Gamma(\beta)\Gamma(4)}\big(\mathrm{B}(2,\beta) - \mathrm{B}(5,\beta)\big), D = \frac{6dc_1}{\Gamma(\beta)\Gamma(5)}\big(\mathrm{B}(2,\beta) - \mathrm{B}(6,\beta)\big),$$

$$E = \frac{a(P-\gamma)}{\Gamma(2)}, F = \frac{b(P-\gamma)}{\Gamma(3)}, G = \frac{2c(P-\gamma)}{\Gamma(4)}, H = \frac{6d(P-\gamma)}{\Gamma(5)}, I = \frac{\gamma a}{\Gamma(\beta+1)},$$

$$J = \frac{\gamma b\mathrm{B}(2,\beta)}{\Gamma(\beta)}, K = C_3.$$

In this case, the inventory model can be written as

$$\begin{cases} Min\;TOC_{1,\beta}(T) = A\,T^{1+\beta} + B\,T^{\beta+2} + C\,T^{\beta+3} + D\,T^{\beta+4} + ET^0 \\ \quad + FT + G\,T^2 + HT^3 + IT^{\beta-1} + JT^{\beta} + KT^{-1} \\ \quad untuk\; T \geq 0 \end{cases} \quad (20)$$

In similar case 1 using primal geometric programming approach, the minimized total average cost and the optimal ordering interval is obtained from (20).

(iv)Case 4: $\boldsymbol{\alpha = 1}$ **and** $\boldsymbol{\beta = 1}$
Here, the total average cost is

$$TOC_{1,1}(T) \quad = \quad A\,T^2 + B\,T^3 + C\,T^4 + D\,T^5 + ET^0 + FT + GT^{-1} \qquad (21)$$

with

$$A = \frac{ac_1}{\Gamma(1)\Gamma(2)}\big(\mathrm{B}(2,1) - \mathrm{B}(3,1)\big) + \frac{2c(P-\gamma)}{\Gamma(4)} + \frac{\gamma c\mathrm{B}(3,1)}{\Gamma(1)},$$

$$B = \frac{bc_1}{\Gamma(1)\Gamma(3)}\big(\mathrm{B}(2,1) - \mathrm{B}(4,1)\big) + \frac{6d(P-\gamma)}{\Gamma(5)} + \frac{\gamma d\mathrm{B}(4,1)}{\Gamma(1)},$$

$$C = \frac{2cc_1}{\Gamma(1)\Gamma(4)}\big(\mathrm{B}(2,1) - \mathrm{B}(5,1)\big), D = \frac{6dc_1}{\Gamma(1)\Gamma(5)}\big(\mathrm{B}(2,1) - \mathrm{B}(6,1)\big),$$

$$E = \frac{a(P-\gamma)}{\Gamma(2)} + \frac{\gamma a}{\Gamma(2)}, F = \frac{b(P-\gamma)}{\Gamma(3)} + \frac{\gamma b\mathrm{B}(2,1)}{\Gamma(1)}, G = C_3.$$

In this case 4, the inventory model can be written as

$$\begin{cases} Min\ TOC_{1,1}(T) \quad = \quad A\,T^2 + B\,T^3 + C\,T^4 + D\,T^5 + ET^0 + FT + GT^{-1} \qquad (22) \\ \qquad Subject\ to\ T \geq 0. \end{cases}$$

In similar case 1 using primal geometric programming approach, the minimized total average cost and the optimal ordering interval is obtained from (22).

IV. Numerical Example

In this section simulated the effect of memory on the inventory model i.e. determining the optimal order interval $T^*_{\alpha,\beta}$ and the minimum average total cost $TOC^*_{\alpha,\beta}$. The simulation was carried out using several α and β values, because α, β is a fractional order with $\alpha, \beta \in (0,1)$, and is given the value of a parameter sourced in [9] as follows:

$$a = 40, b = 20, c = 2, d = 5, C_1 = 15, C_3 = 200, P = 300, \gamma = 0.1$$

Table 1. Minimized total average cost and the optimal ordering interval for $0 < \alpha < 1$ and $\beta = 1$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1	1.4502	1.9069×10^4
0.2	1	1.4042	1.9370×10^4
0.3	1	1.3436	1.9538×10^4
0.4	1	1.2663	$\mathbf{1.9547 \times 10^4}$
0.5	1	1.1689	1.9364×10^4
0.6	1	1.0468	1.8950×10^4
0.7	1	0.8931	1.8253×10^4
0.8	1	0.6993	1.7199×10^4
0.9	1	0.4647	1.5675×10^4
1	1	0.2489	1.3575×10^4

It can be observed from Table 1 that as α memory increases gradually, the minimized average total cost will gradually decrease, i.e. a high advantage occurs in the long memory effect when $\alpha = 0.4$ compared to the short memory effect. On the long memory effect, the minimized average total cost is very low compared to the short or no memory effect. However the optimal ordering interval $T^*_{\alpha,\beta}$ in such cases is different. On the effect of long memory the system takes more time to reach the minimum value of $TOC^*_{\alpha,\beta}$ compared to the memoryless supply system. Therefore the level of sales for long memory has an influence on the system. In achieving the same advantage in the case of a system without memory the store owner must change policies in his business such as service to the public / consumer, store / company environment, product quality, etc.Initially a business starts with a reputation for maximum profit as time goes by the company begins to lose its reputation due to various undesirable causes. In the case of $0 < \alpha < 1$ and $\beta = 1$ the company begins to regress in its business when it reaches a maximum of $\alpha = 0.4$. When it reached critical point, the company began to change its business policy and was careful to restore its reputation back.

Table 2. Minimized total average cost and the optimal ordering interval for $\alpha = 1$and$0 < \beta < 1$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
1	0.1	0.2519	1.3587×10^4
1	0.2	0.2506	1.3589×10^4
1	$\mathbf{0.3}$	0.2497	$\mathbf{1.3589 \times 10^4}$
1	0.4	0.2491	1.3588×10^4
1	0.5	0.2488	1.3586×10^4
1	0.6	0.2486	1.3584×10^4
1	0.7	0.2485	1.3582×10^4
1	0.8	0.2486	1.3579×10^4
1	0.9	0.2487	1.3577×10^4
1	1	0.2489	1.3575×10^4

when the parameters $\alpha = 1$and β (exponent of storage costs) obtained the value of $T^*_{\alpha,\beta}$ and the minimum average total cost $TOC^*_{\alpha,\beta}$ there is no significant difference between the short memory effect and the long memory effect. The critical value of the memory index β occurs at the moment $\beta = 0.3$ i.e. the average total cost minimized to the maximum and then gradually decreases up and down. For the long memory effect of the β parameter, the system does not take longer to reach the minimum value of $TOC^*_{\alpha,\beta}$ compared to the memoryless inventory system. Practically β is a memory parameter that corresponds to the cost of storing inventory. Past experience in this case is considered to be the bad attitude of the shop owner towards the transport driver for the shoe business or fabric business etc.However, in general, transport drivers do not respond to the bad attitude of the owner of the store. On the other hand, the transport driver may give a bad impression as a waiter that is not serious in carrying out his duties. Based on these reasons the system is affected by poor transport services but this is also ineffective.

V. SENSITIVITY ANALYSIS

Furthermore, changes in parameter values will be made to find out what parameters affect the inventory model in determining the value of $T^*_{\alpha,\beta}$and the minimum average total cost $TOC^*_{\alpha,\beta}$which is optimal as follows:

Table 3.Sensitivity analysis on $T^*_{\alpha,\beta}$ and $TOC^*_{\alpha,\beta}$ for $\alpha = 0.1$ and $\beta = 1$

Parameter	Parameter Change (%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$	Parameter	Parameter Change (%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
a	+50%	1.6652	2.3307×10^4	C_1	+50%	1.4294	1.9190×10^4
	+10%	1.4985	1.9961×10^4		+10%	1.4459	1.9094×10^4
	−10%	1.3984	1.8149×10^4		−10%	1.4924	1.8762×10^4
	−50%	1.1374	1.4079×10^4		−50%	1.4731	1.8942×10^4
b	+50%	1.4249	2.2096×10^4	C_3	+50%	1.4545	1.9138×10^4
	+10%	1.4451	1.9675×10^4		+10%	1.4511	1.9083×10^4
	−10%	1.4553	1.8463×10^4		−10%	1.4494	1.9055×10^4

	-50%	1.4761	1.6035×10^4		-50%	1.4459	1.9000×10^4
c	$+50\%$	1.4209	1.9486×10^4	P	$+50\%$	1.4623	2.8409×10^4
	$+10\%$	1.4442	1.9153×10^4		$+10\%$	1.4534	2.0937×10^4
	-10%	1.4562	1.8984×10^4		-10%	1.4464	1.7200×10^4
	-50%	1.4809	1.8642×10^4		-50%	1.4182	0.9724×10^4
d	$+50\%$	1.2945	2.0378×10^4	γ	$+50\%$	1.4501	1.9069×10^4
	$+10\%$	1.4127	1.9358×10^4		$+10\%$	1.4502	1.9069×10^4
	-10%	1.4924	1.8762×10^4		-10%	1.4502	1.9069×10^4
	-50%	1.7370	1.7282×10^4		-50%	1.4503	1.9069×10^4

The minimized total average costs increase with gradual increments of parameters a, b, c, d, C_1, C_3, P. Profits decrease with increasing a, b, c, d, C_1, C_3, P. The changes in $TOC^*_{\alpha,\beta}$ have an impact on the gradual increase of parameters a, P. For γ parameters there is no significant influence on $T^*_{\alpha,\beta}$ and $TOC^*_{\alpha,\beta}$ compared to other parameters so they can be ignored. On the effect of long memory, a, P is a critical parameter of inventory as well as a decision maker.

Table 4.Sensitivity analysis on $T^*_{\alpha,\beta}$ and $TOC^*_{\alpha,\beta}$ for$\alpha = 0.8$and$\beta = 1$

Parameter	Parameter Change (%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$	Parameter	Parameter Change (%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
a	$+50\%$	0.87302	2.3986×10^4	C_1	$+50\%$	0.68932	1.7232×10^4
	$+10\%$	0.73761	1.8580×10^4		$+10\%$	0.69730	1.7206×10^4
	-10%	0.65906	1.5802×10^4		-10%	0.70730	1.7180×10^4
	-50%	0.47551	1.0017×10^4		-50%	0.71005	1.7165×10^4
b	$+50\%$	0.56141	1.8436×10^4	C_3	$+50\%$	0.71959	1.7340×10^4
	$+10\%$	0.66816	1.7464×10^4		$+10\%$	0.70351	1.7227×10^4
	-10%	0.73244	1.6923×10^4		-10%	0.69513	1.7170×10^4
	-50%	0.88324	1.5710×10^4		-50%	0.67761	1.7054×10^4
c	$+50\%$	0.14209	1.9486×10^4	P	$+50\%$	0.69201	2.5622×10^4
	$+10\%$	0.69579	1.7212×10^4		$+10\%$	0.69742	1.8884×10^4
	-10%	0.70298	1.7185×10^4		-10%	0.70166	1.5514×10^4
	-50%	0.71811	1.7130×10^4		-50%	0.71794	0.8774×10^4
d	$+50\%$	0.66556	1.7286×10^4	γ	$+50\%$	0.69929	1.7199×10^4
	$+10\%$	0.69186	1.7217×10^4		$+10\%$	0.69934	1.7199×10^4
	-10%	0.70730	1.7180×10^4		-10%	0.69937	1.7199×10^4
	-50%	0.74476	1.7097×10^4		-50%	0.69942	1.7199×10^4

Changes in parameter values on memory index$\alpha = 0.8$ and $\beta = 1$ obtained optimal ordering intervals reduced by incremental increments of parameter values b, c, d, C_1, P, γ. The optimal ordering interval increases with a gradual increase of parameters a, C_3. The minimized total average costs increase with gradual increments of parameters a, b, c, d, C_1, C_3, P. Profits decrease with increasing a, b, c, d, C_1, C_3, P. Changes in $TOC^*_{\alpha,\beta}$ have an impact on the gradual increase of parameters a, P. For γ parameters there is no significant influence on $T^*_{\alpha,\beta}$ and $TOC^*_{\alpha,\beta}$ compared to other parameters so they can be ignored.

VI. GRAPHICAL PRESENTATION

Graphical presentation of minimized total average cost versus time horizon –T and salvage value per unit using the above numerical example

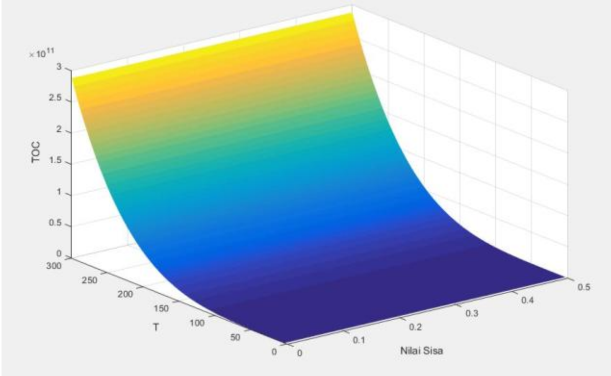

Fig-1Total average cost versus ordering interval-T and salvage value for long memory effect (here $\alpha = 0.1$and$\beta = 1$)

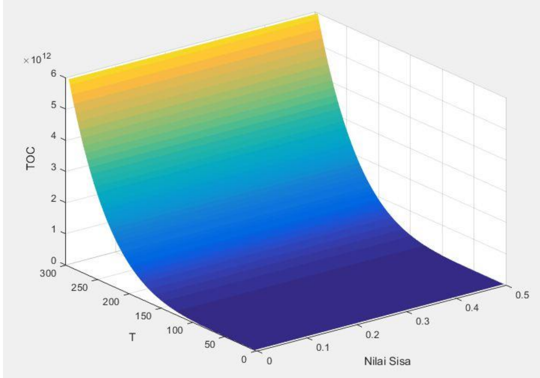

Fig-2Total average cost versus ordering interval-T and salvage value for long memory effect (here $\alpha = 0.9$and$\beta = 1$)

A graph of salvage values, ordering intervals and minimized average total costs shows salvage values have no sensitive effect on long memory or short memory or memoryless systems.

VII.CONCLUSION

The classic EOQ model into a fractional order with a cubic type demand rate and salvage values is essential in determining the optimal ordering interval and the average total cost to be minimized. The importance of the memory effect on the model gives a new direction in inventory management. The α memory index has an important role to show the effect of memory on the inventory model compared to the β memory index. In the long memory effect of $\alpha = 0.4$ the system takes more time to reach the minimum value of the average total cost than the inventory model with the short memory effect. Therefore the inventory model for long memory has an effect on the level of sales. When the α andβ parameters varies the average total cost minimized to a maximum at the time of $\alpha = 0.4$ and $\beta = 0.5$ then gradually decreases up or down.Sensitivity analysis was carried out, namely changes in parameter values on memory index$\alpha = 0.1$ and $\beta = 1$obtained optimal ordering intervals reduced by gradual increases in parameter values b, c, d, C_1, γ. The optimal ordering interval increases with a gradual increase of parameters a, C_3, P . Profits decrease with increases a, b, c, d, C_1, C_3, P. Changes in $TOC^*_{\alpha,\beta}$ have an impact on the gradual increase of parameters a, P while the residual value has no sensitive effect on both long memory and short memory.

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