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A Study on Fractional Derivative of Fractional Power Exponential Function

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ABSTRACT : *Based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication, fractional derivative formula of fractional power exponential function is obtained. This formula is a generalization of the formula in traditional calculus. The chain rule and product rule for fractional derivatives play important roles in this paper. On the other hand, we give some examples to illustrate this formula.*
KEYWORDS: *Jumarie's modified R-L fractional calculus, new multiplication, fractional derivative formula, fractional power exponential function, chain rule, product rule.*

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I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, the application of fractional derivatives and integrals has been scarce until recently. In the last decade, fractional calculus are widely used in physics, mechanics, dynamics, and mathematical economics [1-6]. But the definition of fractional derivative is not unique. Many authors have given the definition of fractional derivative. The commonly used definitions include Riemann-Liouvellie (R-L) fractional derivative, Caputo definition of fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [7-10].

In this article, we obtain the fractional derivative formula of fractional power exponential function based on Jumarie type of R-L fractional calculus. A new multiplication plays an important role in this paper. And the main methods we used are the chain rule and product rule for fractional derivatives. In fact, the formula obtained in this paper is a generalization of the formula in classical calculus. In addition, we give several examples to illustrate this formula.

II. DEFINITIONS AND PROPERTIES

First, we introduce the fractional calculus used in this paper.

Definition 2.1 ([11])**:** Suppose that $0 < \alpha \le 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \quad (1)$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \quad (2)$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([12])**:** *Suppose that* α, β, x_0, C *are real numbers and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \quad (3)$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \quad (4)$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([13])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha((x-x_0)^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$

on some open interval containing x_0, then we say that $f_\alpha((x-x_0)^\alpha)$ is α-fractional analytic at x_0. Moreover, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we define a new multiplication of fractional analytic functions.

Definition 2.4 ([14])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. If $f_\alpha((x-x_0)^\alpha)$ and $g_\alpha((x-x_0)^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha((x-x_0)^\alpha) \otimes g_\alpha((x-x_0)^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha((x-x_0)^\alpha) \otimes g_\alpha((x-x_0)^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5: Suppose that $f_\alpha((x-x_0)^\alpha)$ and $g_\alpha((x-x_0)^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0. If $f_\alpha((x-x_0)^\alpha) \otimes g_\alpha((x-x_0)^\alpha) = 1$, then we say that $g_\alpha((x-x_0)^\alpha)$ is the $\otimes$ reciprocal of $f_\alpha((x-x_0)^\alpha)$, and is denoted by $[f_\alpha((x-x_0)^\alpha)]^{\otimes -1}$.

Definition 2.6 ([15])**:** If $0 < \alpha \le 1$, and let $f_\alpha((x-x_0)^\alpha)$, $g_\alpha((x-x_0)^\alpha)$ be α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}, \tag{9}$$

$$g_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha((x-x_0)^\alpha)$ and $g_\alpha((x-x_0)^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)((x-x_0)^\alpha) = f_\alpha\left(g_\alpha((x-x_0)^\alpha)\right) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha((x-x_0)^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)((x-x_0)^\alpha) = g_\alpha\left(f_\alpha((x-x_0)^\alpha)\right) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha((x-x_0)^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.7 ([15])**:** Let $0 < \alpha \le 1$. If $f_\alpha((x-x_0)^\alpha)$, $g_\alpha((x-x_0)^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)((x-x_0)^\alpha) = (g_\alpha \circ f_\alpha)((x-x_0)^\alpha) = \frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha. \tag{13}$$

Then these two fractional analytic functions are called inverse functions of each other.

The followings are some fractional analytic functions.

Definition 2.8 ([15])**:** If $0 < \alpha \le 1$, and x, x_0 are real numbers. The α-fractional exponential function is defined by

$$E_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(x-x_0)^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \tag{14}$$

The α-fractional logarithmic function $Ln_\alpha((x-x_0)^\alpha)$ is the inverse function of $E_\alpha((x-x_0)^\alpha)$. Furthermore, the α-fractional sine and cosine function are defined as follows:

$$sin_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-x_0)^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{15}$$

and

$$cos_\alpha((x-x_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-x_0)^{2k\alpha}}{\Gamma(2k\alpha+1)}. \tag{16}$$

Moreover,

$$sec_\alpha((x-x_0)^\alpha) = \left(cos_\alpha((x-x_0)^\alpha)\right)^{\otimes -1} \tag{17}$$

is called the α-fractional secant function.

$$csc_\alpha((x-x_0)^\alpha) = \left(sin_\alpha((x-x_0)^\alpha)\right)^{\otimes -1} \tag{18}$$

is the α-fractional cosecant function.

$$tan_\alpha((x-x_0)^\alpha) = sin_\alpha((x-x_0)^\alpha) \otimes sec_\alpha((x-x_0)^\alpha) \tag{19}$$

is the α-fractional tangent function. And

$$cot_\alpha((x-x_0)^\alpha)=cos_\alpha((x-x_0)^\alpha)\otimes csc_\alpha((x-x_0)^\alpha) \tag{20}$$

is the α-fractional cotangent function.

Theorem 2.9 ([16])**:** *Let $0<\alpha\le 1$, then*

$$\left({}_{x_0}D_x^\alpha\right)[sin_\alpha((x-x_0)^\alpha)]=cos_\alpha((x-x_0)^\alpha), \tag{21}$$

$$\left({}_{x_0}D_x^\alpha\right)[cos_\alpha((x-x_0)^\alpha)]=-sin_\alpha((x-x_0)^\alpha), \tag{22}$$

$$\left({}_{x_0}D_x^\alpha\right)[tan_\alpha((x-x_0)^\alpha)]=\left(sec_\alpha((x-x_0)^\alpha)\right)^{\otimes 2}, \tag{23}$$

$$\left({}_{x_0}D_x^\alpha\right)[cot_\alpha((x-x_0)^\alpha)]=-\left(csc_\alpha((x-x_0)^\alpha)\right)^{\otimes 2}, \tag{24}$$

$$\left({}_{x_0}D_x^\alpha\right)[sec_\alpha((x-x_0)^\alpha)]=sec_\alpha((x-x_0)^\alpha)\otimes tan_\alpha((x-x_0)^\alpha), \tag{25}$$

$$\left({}_{x_0}D_x^\alpha\right)[csc_\alpha((x-x_0)^\alpha)]=-csc_\alpha((x-x_0)^\alpha)\otimes cot_\alpha((x-x_0)^\alpha). \tag{26}$$

Definition 2.10: Let $0<\alpha\le 1$. If $u_\alpha((x-x_0)^\alpha)$, $w_\alpha((x-x_0)^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}$ is defined by

$$u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}=E_\alpha\left(w_\alpha((x-x_0)^\alpha)\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\right). \tag{27}$$

Theorem 2.11: (chain rule for fractional derivatives) ([15])**:** *Suppose that $0<\alpha\le 1, x_0$ is a real number, and $f_\alpha((x-x_0)^\alpha), g_\alpha((x-x_0)^\alpha)$ are α-fractional analytic functions. Then*

$$\left({}_{x_0}D_x^\alpha\right)[f_\alpha(g_\alpha((x-x_0)^\alpha))]=\left({}_{x_0}D_x^\alpha\right)[f_\alpha((x-x_0)^\alpha)](g_\alpha((x-x_0)^\alpha))\otimes\left({}_{x_0}D_x^\alpha\right)[g_\alpha((x-x_0)^\alpha)]. \tag{28}$$

Theorem 2.12: (product rule for fractional derivatives) ([16])**:** *If $0<\alpha\le 1, x_0$ is a real number, and assume that $f_\alpha((x-x_0)^\alpha), g_\alpha((x-x_0)^\alpha)$ are α-fractional analytic functions. Then*

$$\begin{aligned}&\left({}_{x_0}D_x^\alpha\right)[f_\alpha(g_\alpha((x-x_0)^\alpha))\otimes g_\alpha((x-x_0)^\alpha)]\\&=\left({}_{x_0}D_x^\alpha\right)[f_\alpha((x-x_0)^\alpha)]\otimes g_\alpha((x-x_0)^\alpha)+f_\alpha((x-x_0)^\alpha)\otimes\left({}_{x_0}D_x^\alpha\right)[g_\alpha((x-x_0)^\alpha)].\end{aligned} \tag{29}$$

III. RESULTS AND EXAMPLES

The following is the fractional derivative formula of fractional power exponential function.

Theorem 3.1: *Assume that $0<\alpha\le 1$, and $u_\alpha((x-x_0)^\alpha)$, $w_\alpha((x-x_0)^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0. Then the α-fractional derivative of the α-fractional power exponential function $u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}$ is*

$$\begin{aligned}&\left({}_{x_0}D_x^\alpha\right)\left[u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}\right]\\&=u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}\otimes\begin{pmatrix}\left({}_{x_0}D_x^\alpha\right)[w_\alpha((x-x_0)^\alpha)]\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\\+w_\alpha((x-x_0)^\alpha)\otimes u_\alpha((x-x_0)^\alpha)^{\otimes -1}\otimes\left({}_{x_0}D_x^\alpha\right)[u_\alpha((x-x_0)^\alpha)]\end{pmatrix}.\end{aligned} \tag{30}$$

Proof By chain rule and product rule for fractional derivatives, we have

$$\begin{aligned}&\left({}_{x_0}D_x^\alpha\right)\left[u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}\right]\\&=\left({}_{x_0}D_x^\alpha\right)\left[E_\alpha\left(w_\alpha((x-x_0)^\alpha)\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\right)\right]\\&=E_\alpha\left(w_\alpha((x-x_0)^\alpha)\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\right)\otimes\left({}_{x_0}D_x^\alpha\right)\left[w_\alpha((x-x_0)^\alpha)\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\right]\\&=u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}\otimes\begin{pmatrix}\left({}_{x_0}D_x^\alpha\right)[w_\alpha((x-x_0)^\alpha)]\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\\+w_\alpha((x-x_0)^\alpha)\otimes\left({}_{x_0}D_x^\alpha\right)\left[Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\right]\end{pmatrix}\\&=u_\alpha((x-x_0)^\alpha)^{\otimes w_\alpha((x-x_0)^\alpha)}\otimes\begin{pmatrix}\left({}_{x_0}D_x^\alpha\right)[w_\alpha((x-x_0)^\alpha)]\otimes Ln_\alpha\left(u_\alpha((x-x_0)^\alpha)\right)\\+w_\alpha((x-x_0)^\alpha)\otimes u_\alpha((x-x_0)^\alpha)^{\otimes -1}\otimes\left({}_{x_0}D_x^\alpha\right)[u_\alpha((x-x_0)^\alpha)]\end{pmatrix}.\end{aligned}$$

Q.e.d.

Next, we give some examples to illustrate the above result.

Example 3.2: Let $0<\alpha\le 1$, then the α-fractional derivative of $cos_\alpha((x-x_0)^\alpha)^{\otimes tan_\alpha((x-x_0)^\alpha)}$ is

$$\begin{aligned}&\left({}_{x_0}D_x^\alpha\right)\left[cos_\alpha((x-x_0)^\alpha)^{\otimes tan_\alpha((x-x_0)^\alpha)}\right]\\&=cos_\alpha((x-x_0)^\alpha)^{\otimes tan_\alpha((x-x_0)^\alpha)}\otimes\begin{pmatrix}\left(sec_\alpha((x-x_0)^\alpha)\right)^{\otimes 2}\otimes Ln_\alpha\left(cos_\alpha((x-x_0)^\alpha)\right)\\+tan_\alpha((x-x_0)^\alpha)\otimes cos_\alpha((x-x_0)^\alpha)^{\otimes -1}\otimes -sin_\alpha((x-x_0)^\alpha)\end{pmatrix}\\&=cos_\alpha((x-x_0)^\alpha)^{\otimes tan_\alpha((x-x_0)^\alpha)}\otimes\begin{pmatrix}\left(sec_\alpha((x-x_0)^\alpha)\right)^{\otimes 2}\otimes Ln_\alpha\left(cos_\alpha((x-x_0)^\alpha)\right)\\-\left(tan_\alpha((x-x_0)^\alpha)\right)^{\otimes 2}\end{pmatrix}.\end{aligned} \tag{31}$$

Example 3.3: Suppose that $0<\alpha\le 1$, and p_α, x_0 are real numbers, then the α-fractional derivative of $p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}$ is

$$\left({}_{x_0}D_x^\alpha\right)\left[p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}\right]=Ln_\alpha(p_\alpha)\cdot p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}. \tag{32}$$

Let $e_\alpha = E_\alpha(1) = 1 + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} + \cdots$, then

$$Ln_\alpha(e_\alpha) = Ln_\alpha\big(E_\alpha(1)\big) = E_\alpha(Ln_\alpha(1)) = E_\alpha(0) = 1. \tag{33}$$

Therefore, by Eq. (32)

$$\left({}_{x_0}D_x^\alpha\right)\left[e_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}\right] = e_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}. \tag{34}$$

Hence,

$$E_\alpha((x-x_0)^\alpha) = e_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha}. \tag{35}$$

Example 3.4: The α-fractional derivative of $\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes sin_\alpha((x-x_0)^\alpha)}$ is

$$\left({}_{x_0}D_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes sin_\alpha((x-x_0)^\alpha)}\right]$$

$$= \left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes sin_\alpha((x-x_0)^\alpha)} \otimes \begin{pmatrix} cos_\alpha((x-x_0)^\alpha) \otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right) \\ +sin_\alpha((x-x_0)^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes -1} \end{pmatrix}. \tag{36}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication, we obtain the fractional derivative formula of fractional power exponential function. This formula is a generalization of the derivative of power exponential function in classical calculus. The main methods we used are the product rule and chain rule for fractional derivatives. In the future, we will continue to use these two methods to study the problems in fractional calculus and fractional differential equations.

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