American Journal of Engineering Research (AJER)
e-ISSN: 2320-0847 p-ISSN : 2320-0936
Volume-11, Issue-03, pp-107-112
www.ajer.org
Research Paper Open Access

The inverse problem and vortex spectrum

José Roberto Mercado-Escalante
10/mar/2022

***Abstract**: The comparison in the description of the flow phenomena with: solutes, commodities and wave numbers, allows us to solve the inverse problem for the fractional diffusion coefficient. We also formulate the alpha-viscosity, the eddy viscosity and their dependencies with the order of the derivatives or spatial occupancy index. The graphs of the spectra separate two regions, those with high wave numbers, dissipative or supplying, and the one with low numbers or demanding, the straight line that separates them represents the eddy viscosity with a constant index. In inverse variables, this spectrum determines a convex curve analogous to the demand and supply curves for goods, similar to the "vis viva" of mechanics and enables its Lagrangian description.*

***Keywords**: Spectra, dissipation and accretion, fractional diffusion coefficient, alpha-viscosity, eddy-viscosity.*

Date of Submission: 02-03-2022 Date of acceptance: 17-03-2022

I. INTRODUCTION

Inverse problems are a very valuable technical tool to discover the rich wealth that is often hidden in the depths, below from the feet of a nation. But on the other hand, as in this paper, they will also allow us to solve problems of theoretical foundations that lead to solve the inverse problem of fractional diffusion and alpha-viscosity.

Throughout the document we return to the Principle of Duality that we observe multi-present in the mythology of the native peoples, in particular in the Goddess Coatlicue, or Mother Goddess, located in the west, representative of running water; in contrast to the God Tlaloc, located in the east, representing the water that falls with the rain.

We can illustrate it with the concept of energy, the potential energy is an energy in latency; it is "vis latens" in contrast to the "vis viva" of kinetic energy.

In the field of fluids, the two variables are: pressure and velocity, as primitive variables. Movement action on field variables is represented by a linear differential operator, which contains the physical parameter of fluid viscosity and the spatial differential operators of divergence and gradient. In its fractional version, explicit reference is doing to fluid movement scales, while in the version called by us “classical” there is no reference to these movement scales. In addition, boundary conditions are associated with an operator that nullifies field variables on said contour, [1].

On the other hand, in the fractional Navier-Stokes equations,fluid movement is described from the Eurelian point of view considering a volume of fluid limited by a boundary surface; with its momentum per unit of volume given by ρV. In first instance, due to its importance, interaction by internal friction is considered. Fractional gradient is expressed by $\nabla^{\beta}_{M}\rho V$, where ρ is the mass density, V the velocity, β the spatial occupation index, and M the measure of mixing of the different spatial directions. Momentum diffusivity is the α -kinematic viscosity ν_{α} , so Darcy's momentum flow is defined by equation (1):

$$q_D = -\nu_{\alpha}\nabla^{\beta}_{M}\rho V (1)$$

Then the rate of momentum change per unit of time is the negative divergence, or convergence, of Darcy's flow, [2]:

$$\frac{d}{dt}\rho V = -\nabla\cdot\left(-\nu_{\alpha}\nabla^{\beta}_{M}\rho V\right) \qquad (2)$$

In second instance, pressure variations contribution to fluid momentum changes through the force that causes the pressure gradient is taken into consideration, in such a way that the composition of the viscous friction stress and the hydrostatic pressure forms the tensor$\rho T = \nu_\alpha \nabla_M^\beta \rho V - \mathrm{pI}$ and gives rise to the deformation law. Thus, the divergence of the stress ρT is the change of momentum $\frac{d}{dt}\rho V$ per unit of volume $\nabla \cdot \rho T = \frac{d}{dt}\rho V$.

Next, an external potential force, per unit volume, of the type $-\nabla\rho\phi$ is incorporated. Next, incompressibility is taken into account. Material derivative that constitutes the local variation with the advective is doing explicit. But the objectivity requirement needs the invariance under coordinate changes, so advective contribution must be modified and the vorticity term arises. Finally, if contribution to the inertial force of the term that contains vorticity is written in the right side of the equation as $\mathrm{V} \times rot\mathrm{V}$, this can be imagine as originated in an external force that dynamizes velocity fields evolution through its vorticity, entering into contradiction with the viscous force, while the other term is interpreted as a restriction that, along current lines, contains the Bernoulli equation:

$$\frac{\partial}{\partial t}\mathrm{V} = \nu_\alpha \nabla_M^\beta V + \mathrm{V} \times \mathrm{rotV} - \nabla\left(\frac{1}{2}(V \cdot V) + \frac{p}{\rho} + \phi\right) \quad (3)$$

Authors also call the coefficient ν_α as the fractional viscosity, due to its units, and it can be compared with the turbulent viscosity of Boussinesq or the eddy-viscosity, and in this research, it will be establishing a relation between alpha-viscosity and eddy-viscosity. Those units are, $[\nu_\alpha] \sim \frac{cm^\alpha}{s}$, so $\left[\nu_\alpha \nabla_M^\beta \rho V\right] \sim \frac{cm}{s^2}$. Therefore, indexed Reynolds number emerges $R_\beta = \frac{ul^\beta}{\nu_\alpha}$, being β the spatial occupation index [1], but the traditional Reynolds number is recovered through the relationship: $R_\beta = \frac{ul}{\nu_\alpha l^{1-\beta}} = \frac{ul}{\nu_2}$, that is, defining the relationship: $\nu_\alpha = \frac{\nu_2}{l^{1-\beta}}$, where the parameter l is understood as the average size of the vortices or the average distances between singularities.

An important approach in hydraulics, particularly in the problem of gauging, is the dynamic Saint-Venant equation, where the deformation tensor is described in equation (4), [3]:

$$\rho \boldsymbol{T} = \left(\frac{\nu_2}{l^{1-\beta}}\right)\nabla^{1+\beta}\rho\boldsymbol{v} - \mathrm{p}\boldsymbol{I} \quad (4)$$

suffers the hydrostatic approach under the hypothesis that gradients are small and/or the length is too large, and is also known as '*gradually varied flow*'. But in addition, a second approach is that of the diffusive wave in Saint-Venant equation, in which it is reduced to the equality of the negative gradient of the water depth with the fractional hydraulic slope. And, in this context, four models for friction slope linked to Authors names are worth mentioning: Chézy, Weisbach-Darcy, Manning and Hagen-Poiseuille [3].

It is an experimental result that the ratio between frictional shear stress (τ_f) and eddy-viscosity (ν_ε) is proportional to the ratio between shear velocity (u_f) and the distance of the boundary $(l_\perp = \mathrm{y})$, in the turbulent motion regime, $\frac{\tau_f}{\rho\nu_\varepsilon} \propto \frac{u_\mathrm{f}}{l_\perp} = \frac{\sqrt{\frac{\tau_0}{\rho}}}{y}$, being $\mathrm{k} = 2.5$ the proportionality constant, ρ the fluid density, τ_0 the same frictional stress at the boundary [4].

We rely on an argument of scales variation and we contract the densities scale until locating us in a fluid of *'ideal gas type'* where viscosity is due to the product of an average velocity by a separation average length between molecules or average free path [4]. If then, in an inverse process, we expand densities scale until we locate ourselves again in a liquid, eddy-viscosity can be understood as the product of the mean velocity (rms, v'_{rms}) by the length of vortices average size $\langle l \rangle$.

II. DIFFUSION

It is known that the change in time of a field can be represented by the anti-gradient of a flow, as in the case of Darcy flow in fluids. A more general notion is the Darcy fractional flux where it is proportional to the fractional anti-gradient of the same field. So, energy of the vortices of a certain wave number evolves by flowing into the vortices of the largest and successive wave number. Darcy flux is proportional to the fractional anti-gradient, of order β, of the energy of a certain level.

However, we want to compare some descriptive aspects of three phenomena: solutions, economics, and turbulence. In the case of solutions, the solute moves in a space of concentrations, from highest to lowest. Regarding demand in the economy, the price curve follows a Pareto distribution and the goods flow in a space of prices, from lowest to highest. In turn, in turbulence, energy flows in a space of scales, from lowest to highest wave number. Thus, on the horizontal, we represent solutes, merchandise, and energy, while on the vertical, we locate concentrations, prices, and wave numbers, [5].

The diffusion operator acts through its coefficient and the field variable, such as $D(\partial/\partial x)$; for goods as $D(\partial/\partial Q)$, and for turbulence $D(\partial/\partial E)$.In the fractional case it is: $D_\beta(\partial/\partial x^\beta), D_\beta(\partial/\partial Q^\beta)$, y $D_\beta(\partial/\partial E^\beta)$.

So, the diffusion equation is: for solutions$(\partial/\partial t)C_o = (-\partial/\partial x)q_D, q_D = D_\beta(-\partial^\beta/\partial x^\beta)C_o$; for goods $(\partial/\partial t)P_d = (-\partial/\partial Q)q_D, q_D = D_\beta(-\partial^\beta/\partial Q^\beta)P_d$; and for turbulence,

$$(\partial/\partial t)k = (-\partial/\partial E)q_D, q_D = D_\beta(-\partial^\beta/\partial E^\beta)k. \qquad (5)$$

We seek solutions of the form$k = (K/K_m) = (E/E_0)^{-e}$; and this is achieved if the diffusion coefficient satisfies, $e^{'}(t) = D_\beta(\Gamma(\beta+1-e)/\Gamma(1-e))$, e < 1, as in the analogous curve- demand.After estimating e′(t), then, the inverse diffusion problem is solved by: $D_\beta = (e)^{'}(t)(\Gamma(1-e)/\Gamma(1-e+\beta)), e < 1, 0 < \beta \leq 1$.(With a convenient clarification, with the index $+\beta$ in the derivative, $-\beta$ should appear in the argument of the function Γ and the graph of the spectrum would be in the second quadrant; what we do is reflect it in the vertical ordinate and place it in the first quadrant).

But it is also possible that the spectral exponent is greater than the unit $e > 1$, as in the case of the spectral exponents: $e = 2, e = 4/3, e = 5/3$ and, in the analogy with the supply curve. Solutions are sought in the form $k = (K/K_m) = (E/E_0)^e$, resulting in $D_\beta = (e)'(t)\ (\Gamma(1+e)/\Gamma(1+\beta+e))$.

On the contrary, for inverse expressions, the variables $(k \leftrightarrow E)$ are exchanged in their graph, making their reflection in the equidistant line constructed with a slope of 45 degrees. Now, the energy evolves in wave number space, or dual space, with a Darcy flux proportional to the fractional anti-gradient of the energy.

$$\frac{\partial}{\partial t}E = -\frac{\partial}{\partial k}q_D, \qquad q_D = D_\beta\left(-\frac{\partial^\beta}{\partial k^\beta}\right)E(6)$$

Figure 1 illustrates two sets of spectra. The first, the highest, is the dissipation region $(4/5) * (\Gamma(1-4/5)/\Gamma(1+x-4/5) + ((5/4) - (4/5))$ and its complement, the inverse cascade $(5/4) * (\Gamma(1+(5/4))/\Gamma(1+x+(5/4)))$. In the second, the dissipative $(3/5) * (\Gamma(1-(3/5))/\Gamma(1-(3/5)+x)) + ((5/3) - (3/5))$ and the inverse cascade $(5/3) * (\Gamma(1+(5/3))/\Gamma(1+(5/3)+x))$. The curves of the regions are shifted up to define the "balance point". The vertical line recalls the domain of interest $0 \leq x \leq 1$.

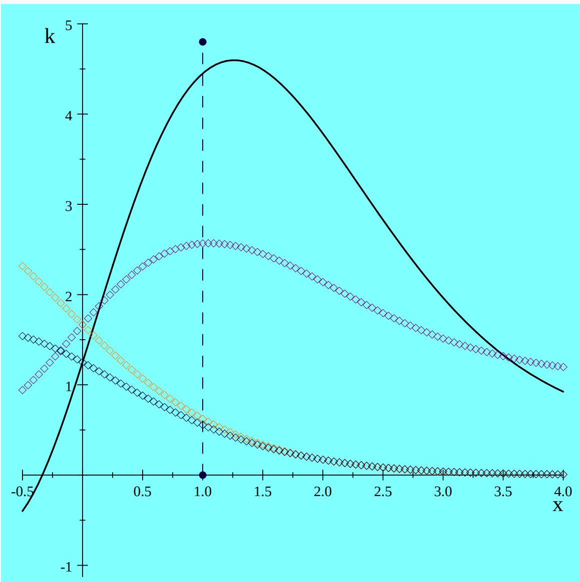

Figure 1. Two sets of spectra, (4/5,5/4) and (3/5,5/3).

III. EDDY-VISCOSITY

According to Boussinesq, 1871, the eddy viscosity is represented by the product of the average rms velocity of the particles by the average size of the eddies or vortices. If we know the speed distribution, we could find the mean rms speed, but in particular the friction speed is one of these, so we think of the rms as proportional to the friction speed, $\nu_\varepsilon = v_{rms} * \langle l \rangle = a * u_f * \langle l \rangle$.
From a second source, the rate of transfer of kinetic energy in the dual space, the cascade process from the vortices with lower wave number to those with higher, is approximated by the ratio between the quadratic speed and the time interval of the considered level, reason why it is given by the quotient between the cubic velocity and the size of the vortices of the aforementioned level; in particular, the transfer rate near the boundary object will follow this same ratio of the cube of the frictional velocity divided by the size of the largest vortices.Consequently, the frictional velocity is expressed by the cube root of the product of the transfer rate and the size of the largest vortices near the boundary, $\varepsilon = u_n^2/\Delta t_n = u_n^3/l_n$, and in particular $u_f = (\varepsilon l_0)^{1/3}$.
Now combining the two sources it would be: $\nu_\varepsilon = a u_f * \langle l \rangle = a(\varepsilon l_0)^{1/3} * \langle l \rangle = a(\varepsilon l_0^4)^{1/3} * \langle l/l_0 \rangle$, or $\nu_\varepsilon = \mathrm{cte}\langle l/l_0 \rangle$.or else in terms of wavenumbers: $\nu_\varepsilon = cte\langle k/k_0 \rangle^{-1}$ with the constant spectral exponent equal to -1. If $\nu_\varepsilon = \nu_2 a(u_f l_0/\nu_2)\langle l/l_0 \rangle = \nu_2 a R_{ef}\langle l/l_0 \rangle$, with $R_{ef} = u_f l_0/\nu_2$, or $\nu_\varepsilon = a(\varepsilon {l_0}^4)^{1/3}\langle l/l_0 \rangle$. Thus, the constant a must be similar in valuetotheVonKármánconstant, $a \approx 0.4 = 1/2.5$.But in addition, thiscould be representedby: $\nu_\varepsilon = cte\langle l/l_0 \rangle^{(\beta+1/3)}\big|_{\beta=2/3} = \mathrm{cte}\langle l/l_0 \rangle^1$.
In Figure 1 and in the position $\beta = e = (4/5) < 1$, we draw a vertical line. We find its inverse image by its reflection in the diagonal inclined 45 degrees, now it is a horizontal line that cuts the axis at $k = (4/5) < 1$, we move it up by the amount $((5/4) - (4/5))$, now cuts the vertical axis at $(4/5) + ((5/4) - (4/5)) = (5/4)$, coinciding with $(5/4) * ((\Gamma(1 + (5/4)))/(\Gamma(1 + x + (5/4))))|_{x=0} = (5/4)$. This is the line that represents the eddy-viscosity.

IV. ALPHA-VISCOSITY

We had defined $\nu_\alpha = \nu_2 l^{\beta-1}$, with the time variable located in ν_2 , now we make it explicit in the first factor, writing $\nu_\alpha = (\nu_2/l^2)l^{\beta+1}$. On the other hand, in the direct cascade we have $l_n = u_n \Delta t_n$ or $(l_n/u_n) = \Delta t_n$, and the transfer rate of energy $\varepsilon \sim u_n^2/\Delta t_n$ or $\Delta t_n \sim u_n^2/\varepsilon$; but since $\Delta t_n \sim l_n^2/\nu_2$, we affirm that $\nu_\alpha = (\varepsilon/u_n^2)l_n^{\beta+1}$, and in particular $\nu_{\alpha 0} = (\varepsilon/u_0^2)l_0^{\beta+1}$; so in dimensionless form it remains: $(\nu_\alpha/\nu_{\alpha 0}) = (u_0^2/u_n^2)(l_n^{\beta+1}/l_0^{\beta+1})$ or $(\nu_\alpha/\nu_{\alpha 0}) = ((\varepsilon l_0)^{2/3}/(\varepsilon l_n)^{2/3})(l_n^{\beta+1}/l_0^{\beta+1})$; that is

$$(\nu_\alpha/\nu_{\alpha 0}) = (l_n/l_0)^{\beta+1/3}. \qquad (7)$$

With $\beta \to 1$, we obtain the Richardson's law in the atmosphere, from 1926, $(l_n/l_0)^{\beta+1/3} \to (l/l_0)^{1+1/3} = (l/l_0)^{4/3}$.
A correspondence can be established between $a = e * (\Gamma(1-e)/\Gamma(1-e+x))$, $e < 1$, and $(a)^{x+1/3}$, as so that given x, there exists a a and a constant C such that $a = C(e * \Gamma(1-e)/\Gamma(1-e+x))^{1/(x+13)}$.

$$e * (\Gamma(1-e)/\Gamma(1-e+x)) \to (a)^{x+1/3} \qquad (8)$$

It can be illustrated with the example of $(4/5) * (\Gamma(1-4/5)/\Gamma(1-(4/5)+\beta)) \to (a)^{\beta+1/3}$, $(1/3 < \beta \leq 4/3)$, but for our purposes, it is only valid until $\beta \leq 1$. In the upper region of the spectrum, it is observed that $a = (k/k_r) > 1$, k_r being the minimum of this upper region; and then, $a = (l_r/l) > 1$ or $l < l_r$, so with increasing wavenumbers, the vortices get smaller and smaller until they reach the dissipative length or Kolmogorov length, $l_d = \eta$ and $\eta^4 = (1/\varepsilon){\nu_2}^3$.
While in the lower region of the spectrum, with increasing β, a decreases, so that a represents k; and since it is less than 1, it will be $a = (k/k_r) < 1$, then $a = (l_r/l) < 1$, being l_r the minimum in this lower region; so, the vortices get bigger and bigger and display a reverse cascade. We can place hurricanes in this accretion region of the spectrum.
On the other hand, according to Chorin, 1988, [7], the spectral exponent must grow with the growth of the index β, $(de/d\beta) > 0$. For a given value of $e > 1$, example (5/4), we draw a horizontal line and find a β_1, now we increase the exponent, example (5/3), and the slice shifts to the right giving a $\beta_2 > \beta_1$.However, in the upper region the opposite happens.We note that the above shift is analogous to "inflation" in economics, where the shift in the direction of price growth leads to a decrease in the quantity consumed. In our context, we will interpret it as less energy consumption by the vortices, and then their growth will slow down. On the other hand, in the dissipation region, the analogy will be with inflation due to a decrease in supply, or scarcity, then the vortices will deliver less energy to the medium, as if the fluid were more viscous. Thus, in Figure 1, we would have $k_{d1} > k_{d2}$, or else $l_{d2} > l_{d1}$.

V. PROBABILITY DENSITIES

In the evolution of wave numbers in energy spaces, these wave numbers flow from the lowest to the highest energy vortices. We see the possibility of this flow between vortices as a Bernoulli random variable, $P(\xi = 1) = p$, $P(\xi = 0) = 1 - p = q$. Then, for n successive independent and homogeneous tests, $\xi_i, i \in 1, n$ of parameter p, where each step occurs with a probability proportional to $p^{i-1}\, q^{n-i}$. We find the constant with the normalization condition and it results, the inverse of a Beta function dependent on the pair of exponents $(i, n - i + 1)$. Then we vary the number of trials and consider the function $f(1 + \beta, n) = B(\beta + 1, n + 1) = \int_0^1 \bar{p}^{\beta} \bar{q}^{n}\, d\bar{p}$, we make the change $\bar{p} = p/n, d\bar{p} = (1/n)dp$ and the evaluation $\bar{p} = 1$ produces $p = n$; then we obtain, $f(1 + \beta, n) = \int_0^n p^{\beta}\, q^{n}\, dp$. Next, we change β to β-e and then we allow the number of trials to be arbitrarily large, so we approach the limit $n \to \infty$ and obtain,

$$f(1 + (\beta - e), n) \xrightarrow{n \to \infty} \Gamma(1 + (\beta - e)) \qquad (9)$$

The above sequence suggests a gamma-like function for the probability density. But our model is predator-prey, in the sense that vortices with larger wavenumbers act as predators and those with smaller numbers as prey, maintaining symmetry. With these elements in mind, we follow a method similar to that of the distribution of Burr, defining a generating function, a function that must be non-negative. We calculate the integral of the generating function and exponentiate by changing the sign. We add the unit and invert, [8].
We obtain a consequence function of the generator: $g(x) \mapsto G_{od}(x) = 1/(1 + exp(-\int_{-\infty}^{x} g(\bar{x})d\bar{x}))$. We choose as the generating function the negative of the digamma function, which is the negative of the logarithmic change of the gamma function: $g(x) = -F_{\Gamma}(x) = -(d\Gamma/\Gamma)(1 + x) = -(d/dx)ln\Gamma(1 + x)$. We integrate and change the sign to obtain the logarithm of the gamma function: $-\int_{-\infty}^{x} g(\bar{x})d\bar{x} = ln\Gamma(1 + x)$. We exponentialize and the gamma function results: $exp(-\int_{-\infty}^{x} g(\bar{x})d\bar{x}) = \Gamma(x + 1)$. We add the unit: $1 + \Gamma(x + 1)$. We invert: $G_{od}(x) = (1/(1 + \Gamma(1 + x)))$. Finally, we shift the domain of x and perform a convolution product centered at $x = 3/2$. The result is $G_d(x) = (1/(1 + \Gamma(x)))(1/(1 + \Gamma(3 - x)))$ and its plot shows a mound-like appearance similar to the predator- prey. Its maximum in 3/2 takes the value $(1/((1/2)\sqrt{\pi} + 1)^2) = 0.28107$ and the area that normalizes the mound is 0.55963.
To verify the non-negativity of the generator, we see that the digamma function is between two logarithmic: $log(x) - (1/x) \le F_{\Gamma}(x) \le log(x) - (1/2x)$. We check the null value of the upper one: $log(x) - (1/2x) = 0$, or $xlog\, x = (1/2)$, we invert, and we obtain the Lambert function evaluated in (1/2), $W(1/2) = 0.35173$, so $x = e^{W(1/2)}$, $x = 1.4215$. The domain of the digamma is $x = \beta - e$, which does not reach the value 1.4215. Therefore, in the considered domain the non-negativity of the generating function is ensured.
Figure 2 shows the probability density associated with the dissipation region, $(1/(1 + \Gamma(x)))(1/(1 + \Gamma(3 - x)))$, a mound centered at 3/2, of height $1/((1/2)\sqrt{\pi} + 1)^2)$, but normalized by the area 0.55963.

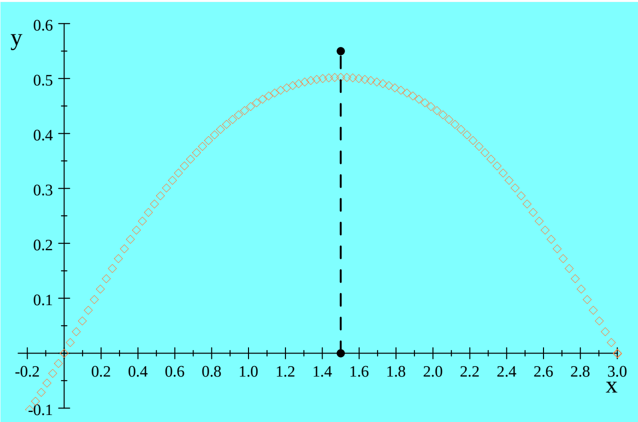

Figure 2. The probability density associated with the dissipation region.

VI. CONCLUSIONS

- The solutions of the inverse problem, in direct variables, demarcate two regions, the upper one with relatively large wavenumbers and the lower one with small numbers. The upper one has the greatest curvature and is bounded from above, while the lower one shows less curvature and is bounded by zero. The horizontal line separating the two regions represents the Boussinesq eddy viscosity.

- In inverse variables, a region delimited by a convex curve is determined, with a lower saddle point where they intersect. The curves allow a Lagrangian description. The saddle point exemplifies the Hurwicz Saddle Point Theorem equivalent to the Hahn-Banach Theorem, considered one of the pillars of Functional Analysis, [9]. The vertical line from the saddle point represents the eddy viscosity and is the economic analog of zero elasticity.
- The upper region could be called the dissipative or supplying region, because it offers energy to the middle similar to a supply curve in the economic field. While the lower one would be the demanding one, which requires the energy of the medium to grow in size.
- The higher the spectral index, the greater the slope with respect to the order of the derivative or occupancy index β, in the upper region where $e < 1$. Conversely, in the lower region, the curve linked to the one with the highest spectral index decreases more slowly, and the opposite for the one with the lowest index.
- The lower curve corresponds to a Pareto distribution, with an exponent greater than 1, in absolute value. The lower the value of the exponent, the more horizontal the curve is, at a limit of 1, it is a horizontal line that represents the eddy viscosity.
- The probability density of the dissipation region is described by a symmetric and finite mound, built on the basis of the gamma function and analogous to the logistic density of the predator-prey model.

REFERENCES

[1]. J. R. Mercado-Escalante, P. A. Guido-Aldana, Fractional visión ofthe fluid-particleinteraction and itsmodels. International JournalofEngineeringResearch and Applications, www.ijera.com, ISSN: 2248-9622, Vol. 8, Issue 7, (Part-V), July 2018, pp. 24-32. DOI: 10.9790/9622-0807052432.

[2]. J. R. Mercado-Escalante, M. Íñiguez-Covarrubias, P. A. Guido-Aldana, Navier-Stokes fractionalequations and streamsgauging. International Journal of EngineeringResearch and Applications, www.ijera.com, ISSN: 2248-9622, Vol. 9, Issue 2, (Series-III), Feb 2019, pp. 01-07. DOI: 10.9790/9622-0902030107.

[3]. J. R. Mercado, A. Aldama y F. Brambila, Un problema inverso en las ecuaciones de Saint-Venant. (An inverse problem in the Saint-Venant equations). AportacionesMatemáticas, Serie Comunicaciones. 32, pp. 3-30, 2003.

[4]. H. Rouse, Elementary Mechanics of Fluids, Dover, Publ., New York, (1946). pp. 376.

[5]. J. R. Mercado-Escalante, Algebra of saving, International Journal of Engineering Research and Applications, www.ijera.com, ISSN: 2248-9622, Vol. 11, Issue 11, (Series-II) November 2021, pp. 50-59.

[6]. H. A Rose, P. L Sulem, Fully Developed Turbulence and Statistical Mechanics. Le Journal de Physique, Tome 39, N. 5, Mai 1978, pp. 441-484.

[7]. A. J. Chorin, Spectrum, Dimension, and Polymer Analogies in Fluid Turbulence. Physical Review Letters, Vol. 60, Number 19, 9 May 1988, pp. 1947-1949.

[8]. V. S. Koroliuk, Manual de la teoría de probabilidades y estadísticamatemática, Ed. Mir, Moscú, (1981), pp. 580.

[9]. R. B. Holmes, Geometric Functional Analysis and its Applications, Springer-Verlag, 1975.

AUTHOR BIOGRAPHY

José Roberto Mercado-Escalante. Independent Researcher, Mexico. Dr. of Sciences (Mathematics) by the National Autonomous University of Mexico-UNAM, Master Degree in Mathematics by the Autonomous University of Puebla-BUAP, graduates in Physics by the National University of Colombia. Research areas: inverse problems, fractals and fractional derivatives, physical and mathematical aspects of hydraulics. E-mail: jrmercadoe@yahoo.com