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Non-Harmonic Oscillations of A System of Three Electric Charges

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Abstract. *In this paper, we consider the problem of three bodies. The problem of calculating the oscillation period of a system of three identical electric charges is solved. The following problem is considered: three identical small balls of the same mass, connected by silk threads of the same length, have the same electric charge and form a regular triangle. Gravity is absent. One of the threads burned out. Determine the period of arising oscillations. Differential equations are obtained that describe the motion of charges. These equations are solved numerically by Euler method. MS Excel spreadsheets are used as a programming environment, the program is written in VBA (Visual Basic for Application).*

Key words: *closed system, law of conservation of energy, Coulomb repulsion, differential equations of motion, improved Euler's method.*

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I. Introduction

Nowadays, more than one serious scientific problem, in which it is necessary to obtain a specific numerical result, cannot be solved without the use of a computer. Computer modeling of physical processes and setting up computer experiments have become an integral part of the study of physics. At present, along with experimental and theoretical physics, a new name has even appeared "computer physics".

The solution of many physical problems is reduced to the solution of differential equations describing the considered physical process. Analytical solutions to these equations are often difficult to find. In many cases, it is easier to find numerical solutions to these equations using a computer. As our teaching experience shows, numerical modeling of physical processes develops a deeper understanding of physics. The sooner students start using the computer for solving physical problems, the more benefits it will bring in their future professional activities.

In this paper, we solve an interesting problem, the analytical solution of which leads to elliptic integrals. A differential equation is obtained that describes the oscillation of electric charges. This equation is solved numerically by the improved Euler method [1–3]. MS Excel spreadsheets are used as a programming environment. The program is written in VBA (Visual Basic for Application).

So let's consider the following problem. Three identical small balls of mass, connected by silk threads of length , have the same electric charge and form a regular triangle. There is no gravity. One of the threads is burned out. Determine the period of arising oscillations.

Solution

In the considered system of charges, only internal forces act. These forces cannot change the position of the center of mass of the system, so the position of the center of mass is preserved. The center of mass of an equilateral triangle is located at the point of intersection of the medians, which coincides with the center of an equilateral triangle. We choose the origin of the Cartesian coordinate system at the center of mass, pointing the axis vertically upwards, and the axis horizontally, as shown in Fig. 1.

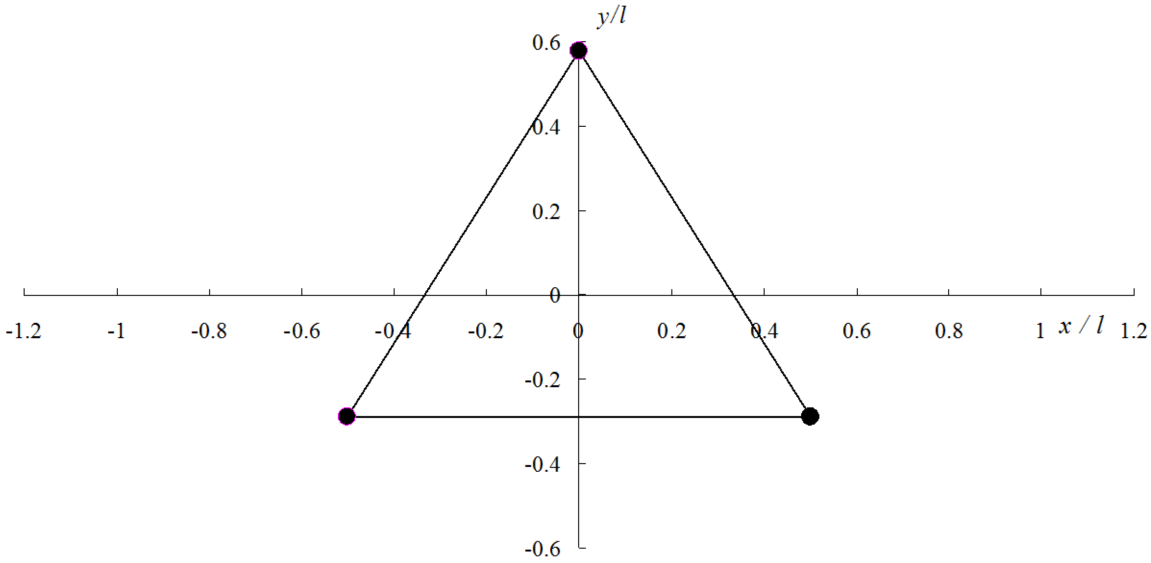

Fig. 1 – Three identical charges connected by threads of length l.

After the burnout of the horizontal thread, the forces acting between the central and extreme charges do not change, and the oscillations arise due to the action of the Coulomb repulsive forces between the extreme charges. Due to symmetry, the central charge will oscillate along the axis y. Thus, this mechanical system has one degree of freedom, which we will describe by the coordinate y_1 of the central charge. Since this system of charges is closed, its total energy E is conserved:

$$E = \frac{q^2}{4\pi\varepsilon_0 l} \,. \tag{1}$$

We write the law of conservation of energy in the form:

$$m\dot{y}_1^2 + 2m\left[\left(\frac{\dot{y}_1}{2}\right)^2 + \dot{x}_2^2\right] + \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2x_2} = E \,. \tag{2}$$

Here, x_2 is the x-coordinate of the rightmost charge and it is taken into account that in the center-of-mass system the y-coordinate of the outermost charges is 2 times smaller in absolute value than the coordinate y_1 of the central charge: $y_2 = -\frac{y_1}{2}$. Since the position of the center of mass of the system does not change during the movement, the relation must be satisfied:

$$\left(y_1 + \frac{y_1}{2}\right)^2 + x_2^2 = l^2 \,. \tag{3}$$

Taking into account (3), we write the law of conservation of energy (2) in the form:

$$\frac{3}{4}\dot{y}_1 + \dot{x}_2^2 = \frac{q^2}{8\pi\varepsilon_0 m}\left(\frac{2}{l} - \frac{1}{x_2}\right). \tag{4}$$

We will measure the coordinates of charges y_1 and x_2 in units l and introduce dimensionless time τ :

$$\tau = \frac{t}{\tau_0}, \tag{5}$$

where

$$\tau_0 = \frac{l^{\frac{3}{2}}}{q}\sqrt{8\pi\varepsilon_0 m}\,. \tag{6}$$

We write expressions (3) and (4) in dimensionless variables:

$$\left(\frac{3y_1}{2}\right)^2 + x_2^2 = 1, \tag{7}$$

$$\frac{3}{4}\dot{y}_1^2 + \dot{x}_2^2 = 2 - \frac{1}{x_2}\,. \tag{8}$$

Here, the dots now denote the derivatives with respect to the dimensionless time τ .
From relation (7) we find:

$$x_2 = \sqrt{1-\left(\frac{3y_1}{2}\right)^2}, \tag{9}$$

$$\dot{x}_2^2 = \left(\frac{3}{2}\right)^4 \frac{y_1^2\dot{y}_1^2}{1-\left(\frac{3y_1}{2}\right)^2}. \tag{10}$$

Substituting (9) and (10) into (8), we get:

$$\dot{y}_1^2 = \frac{\left(2\sqrt{1-\left(\frac{3y_1}{2}\right)^2}-1\right)\sqrt{1-\left(\frac{3y_1}{2}\right)^2}}{\frac{3}{4}\left(1+\frac{9}{2}y_1^2\right)}. \tag{11}$$

At the initial moment $y_1 = \frac{1}{\sqrt{3}}$ and $\dot{y}_1 = 0$, and at $y_1 = 0$, we get $\dot{y}_1 = \mp\frac{2}{\sqrt{3}}$. The resulting relation (11) allows one to construct the phase diagram of oscillations, which is shown in Fig. 2.

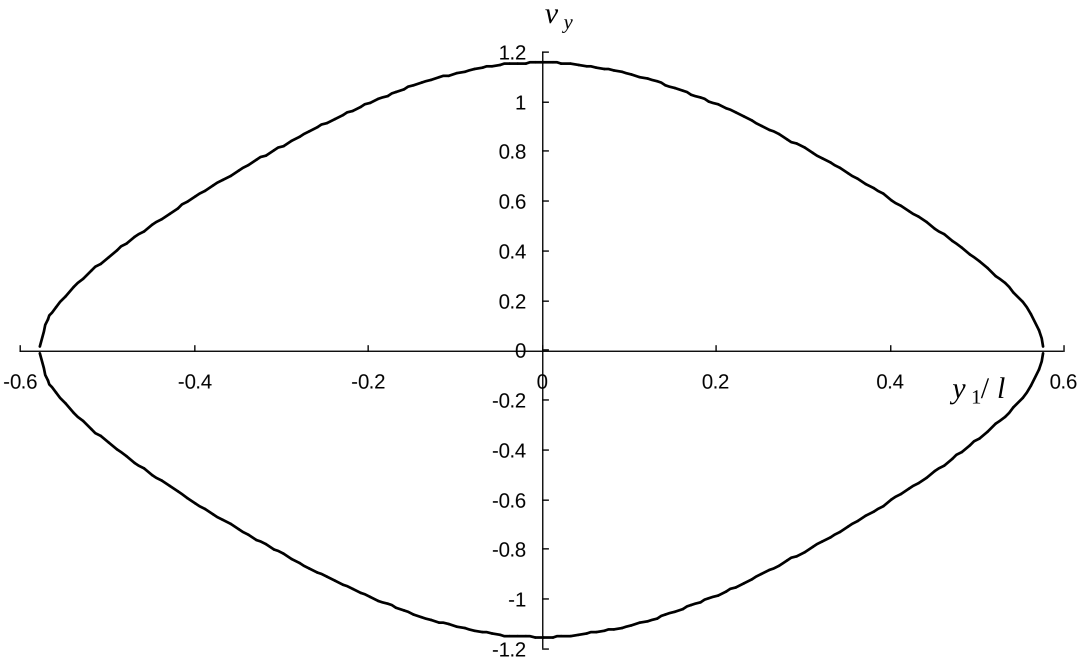

Fig. 2 – Phase diagram of oscillations

Here, the dimensionless velocity of the central charge is given by:

$$v_y = \frac{dy_1}{d\tau} = \dot{y}_1. \tag{12}$$

Equation (11) is not integrated in elementary functions. Its solution can be written in terms of elliptic integrals [5-8]. However, equation (11) can be easily integrated numerically, for example, by the improved Euler method. The dependence of the dimensionless coordinate obtained as a result of numerical integration is $y_1(\tau)$ shown in Fig. 3. From the obtained dependence, the period of arising oscillations can be determined.

Using expression (12), we can write the following expression for the oscillation period:

$$T = 4\tau_0 \int_0^{1/\sqrt{3}} \frac{dy_1}{|\dot{y}_1|}. \tag{13}$$

Numerically calculating the integral in (13), we obtain:

$$\int_0^{1/\sqrt{3}} \frac{dy_1}{|\dot{y}_1|} = \int_0^{1/\sqrt{3}} \frac{\frac{3}{4}\left(1+\frac{9}{2}y_1^2\right)dy_1}{\sqrt{\left[2\sqrt{1-\left(\frac{3}{2}y_1\right)^2}-1\right]\sqrt{1-\left(\frac{3}{2}y_1\right)^2}}} \cong 1.095. \tag{14}$$

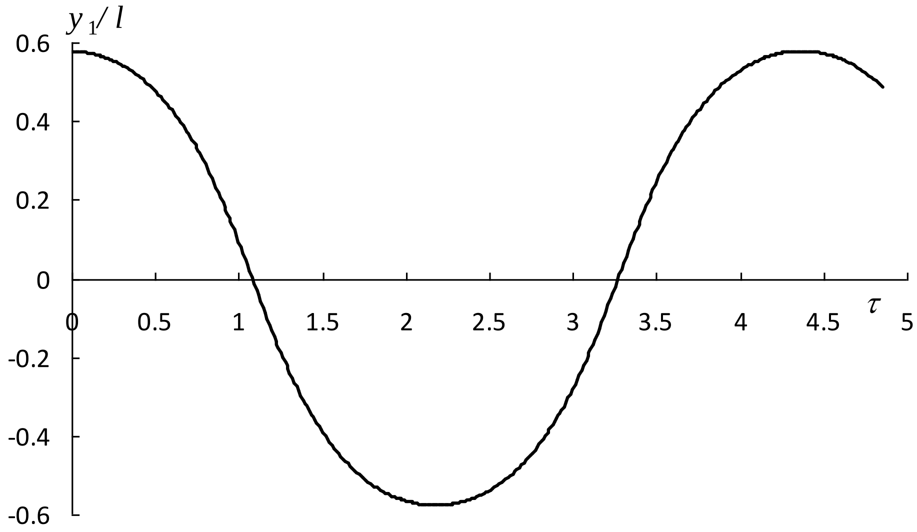

Fig. 3 – Dependence of the *y*-coordinate of the central charge on the dimensionless time τ

Substituting the calculated value of the integral into (13), we find that in units τ, the period is approximately 4.38. Thus, the following expression can be written for the oscillation period:

$$T \cong 4.38\,\frac{l^{\frac{3}{2}}}{q}\sqrt{8\pi\varepsilon_0 m}\,. \tag{15}$$

Differentiating expression (11) with respect to dimensionless time τ, we find the acceleration with which the central charge moves:

$$\ddot{y}_1 = -\frac{f'}{2f}\dot{y}_1 - \frac{9y_1}{8f\left[1-\left(\frac{3}{2}y_1\right)^2\right]^{\frac{3}{2}}}. \tag{16}$$

Here

$$f \equiv f(y_1) = \frac{3}{4} + \left(\frac{3}{2}\right)^4 \frac{y_1^2}{1-\left(\frac{3}{2}y_1\right)^2}, \tag{17}$$

$$f' = \frac{df}{dy_1} = \left(\frac{3}{2}\right)^4 \frac{2y_1}{\left[1-\left(\frac{3}{2}y_1\right)^2\right]^2}. \tag{18}$$

So, at the initial moment of time, immediately after the thread is burned out: $y_1 = \frac{1}{\sqrt{3}}$, $\dot{y}_1 = 0$, we get $\ddot{y}_1 = \ddot{y}_0 = -\frac{4\sqrt{3}}{10}$. In dimensional variables, the initial acceleration is determined by the expression:

$$a_0 = \frac{l}{\tau_0^2}\ddot{y}_0 = -\frac{q^2\sqrt{3}}{20\pi\varepsilon_0 l^2 m}. \tag{19}$$

The last expression gives the answer to problem 2.6 from Irodov's problem book [4].

We also note that to simulate charge oscillations, it is convenient to integrate numerically not equation (11), but equation (16), since the signs of the variables automatically change in this case and it is better to use it to visualize oscillations. We present here the algorithm of the improved Euler method for integrating equation (16):

$$\begin{aligned} y_{n+1} &= y_n + \dot{y}_{n+\frac{1}{2}} h, \\ \dot{y}_{n+\frac{1}{2}} &= \dot{y}_{n-\frac{1}{2}} + \ddot{y}_n h, \\ y_{n+\frac{1}{2}} &= y_{n-\frac{1}{2}} + \dot{y}_n h, \\ \dot{y}_{n+1} &= \dot{y}_{n-1} + \ddot{y}_{n+\frac{1}{2}} h. \end{aligned} \tag{20}$$

Here the index "1" of the variable y_1 is omitted; $n = 0, 1, 2, 3, \ldots$; $\tau_n = nh$; $y_0 = \frac{1}{\sqrt{3}}$; $\dot{y}_0 = 0$; $\dot{y}_{\frac{1}{2}} = \ddot{y}_n h/2$; $y_{\frac{1}{2}} = y_0 + \dot{y}_{\frac{1}{2}} h/4$, where h is the integration step.

II. Conclusion

In this article, we wanted to show the advantages of introducing dimensionless variables, which simplifies numerical calculations and makes the results more universal.

It follows from expression (14) that the square of the period is proportional to the cube of the linear dimensions of the system: $T^2 \approx l^3$, such a dependence is typical for all fields of the Coulomb type. In such force fields, the interaction force of point bodies is inversely proportional to the square of the distance between the bodies, and the potential energy is inversely proportional to the distance between the bodies. In particular, this dependence of force on distance leads to Kepler's third law for the motion of planets in the gravitational field of the Sun.

This problem may also be of interest when considering oscillations of quarks in baryons, where oscillations can arise due to the electrical repulsion of two identical quarks.

To calculate the integral (14), one can use the Monte Carlo method and the mean value theorem:

$$S = \int_a^b f(x)\,dx = (b-a)\langle f(x)\rangle. \tag{21}$$

Here, the mean value of the function over a given interval can be found using the Monte Carlo method:

$$\langle f(x)\rangle = \frac{1}{N}\sum_{i=1}^{N} f(x_i), \tag{22}$$

where x_i is determined using a random number generator:

$$x_i = a + (b-a)\,\mathrm{Rnd}(). \tag{23}$$

Here Rnd() is the standard program. Assuming $a=0$, $b=\frac{1}{\sqrt{3}}$ and defining the integrand function by expression (14), we calculate the value of the integral in (14). Below is a listing of the Period program is written VBA in MS Excel that allows you to perform this calculation.

```
Sub Period()
Sheets("Int").Select

aa1 = Cells(2, 2)
bb1 = Cells(3, 2)
NN = Cells(4, 2)

ab = aa1 - bb1
ss = 0

For jj = 1 To NN
yy = ab * Rnd()
ff = Sqr(3 * (1 + 18 * yy * yy / 4) / 4) / Sqr((2 * Sqr(1 - (3 * yy / 2) * (3 * yy / 2)) - 1) * Sqr(1 - (3 * yy / 2) * (3
* yy / 2)))
ss = ss + ff
Next jj

ss = ss / NN
Cells(7, 2) = ss * ab
End Sub
```

The relative error in the calculation of the integral δ is determined by the reciprocal root law. So with the value of the variable NN= $N=10^6$ we get $\delta\cong\frac{1}{\sqrt{N}}=10^{-3}$.

Acknowledgments

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