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# Canal surfaces and its application to the CAGD

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***Abstract***
*This paper deals with Canal surfaces, a particular kind of surfaces which have rational parametrizations. We base our study on the con- cept of focal sets, which is required to characterize these surfaces, and we describe the three most representative types of these surfaces: tubular surfaces, Dupin cyclides and surfaces of revolution. The main goals of the paper are to obtain new properties of this type of surfaces and to show a novel application of them to computer-aided design, particularly in Engineering and Architecture.*
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***Keywords**: Canal surfaces; pipe surfaces; Dupin cyclides; computer-aided de- sign.*
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## I. INTRODUCTION

In the last five years several papers on different types of non-usual surfaces ap- pearing in Engineering have been published. For instance, Burak et al. dealt in 2016 with non-null cylindrical surfaces in [14], Yariv and Schnitzer with superhy- drophobic surfaces of closely spaced circular bubbles in 2018 [17] and Yang et al. in 2019 with curved surfaces in 3D [16]. According to this research we deal in this paper with another particular type of surfaces, the Canal surfaces. Our objective is to show that these surfaces could be used in Architecture and Engineering to resolve certain problems, for instance, in Mechanics Engineering to make the de- sign of simple pieces or assemblies (mechanisms) in which the dimensions must be precise.

The first author who deeply studied Canal Surfaces was the French mathe- matician Gaspard Monge, who gave them this name due to its aspect. Later, its study was carried out by well-known geometers as Dupin and several physicists, like Maxwell, for instance. These surfaces generalize other well-known ones, the surfaces of revolution.

In a no formal way, a Canal surface is defined as an envelope of spheres, which can be of no constant radius, whose centers are placed in a curve (see Figure 1.1). In any case, a precise definition of such object will be showed later.

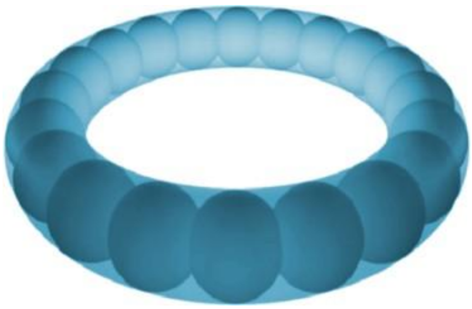

Figure 1.1: A Canal surface as an envelope of spheres

One could think in principle that these surfaces have their maximum interest and treatment in Differential Geometry, although the reality is quite different. In- deed, its main interest lies in the field of its practical applications, specifically in CAGD (Computer Aided Geometric Design), where lots of studies of surface con- structions are carried out. These studies allow deformations (blending surfaces), reconstruction of shapes, transition surfaces among tubes, planning of robot move- ments, etc, (see [5, 12, 11], for instance). In many of these works the so-called Pipe Surfaces are considered (see Figure 1.2). These are Canal surfaces with a constant radio, which have many practical applications. For example, they may represent the surface of a spring (ideal helix), which we can see in the real life in climbing plants, corkscrews, etc., because such images can be approximated by pieces of tori attached at a certain angle between their successive axes. They might even have pieces of circular cylinders.

Within the Canal surfaces, we emphasize the *blending surfaces*. They are obtained from an initial Canal surface by an operation consisting of generating one or several auxiliary surfaces that create a differentiable transition among them, in the way that the object finally obtained is the union of all the previous surfaces in a simple piece. These last surfaces are called *primary surfaces.* Many of the objects that can be found in Engineering are elementary surfaces (as cylinders, cones, tori, spheres and planes), although for blending surfaces it is necessary to work with free-form surfaces (see [3, 9]).

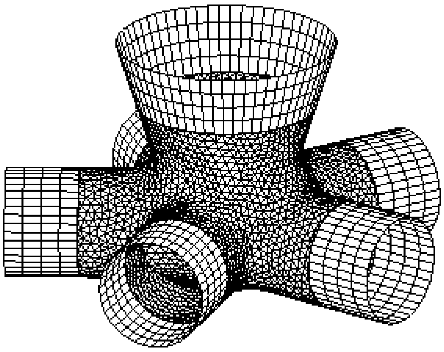

**Figure 1.2:** Pipe Surface

The structure of the paper is as follows: After this Introduction, we show in Section 2 a historical evolution of CAGD (Computer Aided Geometric Design), with the objective of a better understanding of the proposal of the paper. In Section 3 we recall some preliminaries on Differential Geometry of curves and surfaces. Section 4 deals with the focal set, required to characterize Canal surfaces. The main characteristics and the different types of these Canal Surfaces are shown in Section 5, in which new results on them have been also introduced. Finally, in Section 6, some applications of Canal surfaces to CAGD are commented, partic- ularly, we present examples made by using the Blender program (so called for constructing a particular case of Canal surfaces, the blending surfaces), whose potential is enormous: in addition to modeling, it can be used to make special effects, develop video-games or even create animations.

## II. A BRIEF HISTORICAL EVOLUTION OF CAGD

This section shows a very brief historical overview of the main developments related with curves and surfaces when they appeared in the field of *CAGD - Com- puter Aided Geometric Design -* until the middle 1980s.

CAGD was introduced due to the influence of different areas in the 1950s and 1960s, although it is also certain that interactions with different fields of science and engineering were not limited to those years. It is assumed that CAGD deals with the construction and representation of free-form curves, surfaces, or volumes. The reader can find a much more detailed and complete information of this subject in [2], from which the vast majority of the following historical data have been obtained.

R. Barnhill and R. Riesenfeld were the first authors who used the term *CAGD* when they organized a conference on that subject at the University of Utah in 1974. That event, in which lot of researchers from the U.S.A. and Europe par- ticipated, might be considered the founding event of the field. I. Faux and M. Pratt [6] were the authors of the first textbook on that topic, *Computational Ge- ometry for Design and Manufacture*, in 1979 (note that the meaning of that term *Computational Geometry* has quite changed since then. It is used to describe a discipline mostly dealing with discrete geometry, related with the complexity of algorithms). R. Barnhill and W. Boehm took advantage of it to found the journal *Computer Aided Geometric Design* in 1984. In any case, the principal text for thissubject is the one by Preparata and Shamos [13].

After Utah, a new conference,organized by P. Bezier, who was president of the Societe des Ingenieurs de I'Automobile in that time was held in Paris, in 1971.

It is convenient to recall that in AD Roman times, people already use curves in a manufacturing environment, with the objective of shipbuilding.This use was later perfected by the Venetians from the 13th to the 16th century. In that epoch, drawings to define a ship hull became popular in the 1600s in England.

With the advent of computers, three centuries later, R. Liming, who had worked for the North American Aviation during World War II, published in 1944 the book *Analytical Geometry with Application to Aircraft*, in which he combined classical drafting methods with computational techniques for the first time. It is really recognized that computers were totally necessary for disciplines such as CAGD to emerge. One of the first companies to use computational techniques was General Motors, which developed its DAC-I (Design Augmented by Computer). It used curve and surface techniques developed by several researchers, C. de Boor and W. Gordon between them. Plotting, or drafting, was used to the main automotive brands of the moment, such as Citroen or its competitor, Renault, both located in Paris. Note that, during the early 1960s, the previously mentioned Pierre Bezier, who was the French engineer who introduced the so-called Bézier curves and sur-faces, headed the design department of the last one.

Several branches of Mathematics, such as the differential geometry of para- metric curves, the approximation theory and numerical analysis were combined to become important building blocks of CAGD. Indeed, the US aircraft company Boeing employed them around 1950 to obtain improvements in different parts of the planes, as software based on Liming's conic constructions when design- ing airplane fuselages or the use of *spline curves* for the design of wings (*B- splines curves* were introduced by I. Schoenberg in 1946 for the case of uniform knots. They are very in approximation theory). *Parametric surfaces*, introduced by Gauss and Euler, were also adopted in early CAD (computer-aided design) de-velopments, for instance, to trace a surface for plotting or for driving a milling tool. At the end of 1950s, these surfaces were studied and used by several compa-nies in Europe and the U.S., as Boeing or General Motors.

With respect to the scientific applications of CAGD, lot of them can be men-tioned. It is well-known that many scientific disciplines need to model phenomena for which only a set of discrete measurements is available but a continuous modelis also desired.

For instance, with respect to *data sites*, which are typically 2D, but might also be 3D and whose location has no structure, was introduced a function, the *scattered data interpolant*, which interpolates the given data values and gives rea-sonable estimates in between. One approach to it was made by R. Hardy in 1971, who generalized the concept of splines to surfaces and R. Sibson developed a scheme, the *nearest neighbor interpolation*, which was based on the concept of Voronoi diagrams, also known as Dirichlet tessellations, a concept very related with Computational Geometry. Moreover, another relevant connection between CAGD and this last type of geometry is that of triangulation algorithms (the goal is to find a set of triangles having a given 2D point set as vertices). C. Lawson published in 1971 the first algorithm and Green and Sibson constructed an algo- rithmic connection between triangulations and Voronoi diagrams.

Although in a very summarized way, all of the above shows that CAGD needs and relies on different branches of Mathematics and that CAGD and Computer Graphics need each other. In the following sections of this work we will deal with the way in which Differential Geometry could also help in the use of these new techniques.

## III. Preliminaries

In this paper we will deal with several well-known concepts on Differential Geometry, particularly curves and surfaces. We recall here some of them, taking in mind that the reader can consult [1] for further information.

A *regular parametrized curve* in $\mathbb{R}^m$ is a mapping $\alpha : (a,b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m : t \mapsto \alpha(t) = (x_1(t), ..., x_m(t))$ verifying the two following conditions: Differentiability Condition: $\alpha \in \mathbb{C}^k, k \geq 1$ and Regularity Condition: $\alpha'(t) = \dfrac{d\alpha}{dt} \neq 0, \forall t \in (a,b)$.

The triple $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ formed by the tangent, main normal and binormal unitary vectors of the curve $\alpha(s)$ is called *Triple* or *Frenet Reference* in the point $\alpha(s)$ of $\alpha$.

Moving on surfaces, we recall that a *simple surface* or *local chart* is a mapping $\mathbb{X} : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ injective and infinitely differentiable and if $\mathbb{X} = \mathbb{X}(u,v)$, then $\dfrac{\partial \mathbb{X}}{\partial u} \times \dfrac{\partial \mathbb{X}}{\partial v} \neq 0, \forall (u, v) \in U.$

The point set $M \subset \mathbb{R}^3$ is said to be a *regular surface* if there exists a family of simple surfaces $\mathcal{A} = \{\mathbb{X}_i : U_i \longrightarrow \mathbb{R}^3, i \in I\}$ recovering it. Such a family $\mathcal{A}$ is named an *atlas* of M.

The two *principal curvatures* at a given point of a simple surface, denoted by $\kappa_1$ and $\kappa_2$, are the eigenvalues of the shape operator at the point. They measure how the surface bends by different amounts in different directions at that point.

A surface generated by rotating a two-dimensional regular parametrized curve $\alpha : (a, b) \longrightarrow \mathbb{R}^3, \alpha(t) = (x(t), 0, z(t))$ in the oxz-plane, with $x(t) > 0$ is called a *surface of revolution*. It always has azimuthal symmetry.

Examples of surfaces of revolution include (in the alphabetical order) the apple, cone (excluding the base), conical frustum (excluding the ends), cylinder (excluding the ends), Darwin-de Sitter spheroid, Gabriel's horn, hyperboloid, lemon, oblate spheroid, paraboloid, prolate spheroid, pseudo-sphere, sphere, spheroid, and torus (and its generalization, the toroid).

The standard parametrization of a surface of revolution as a simple surface is

$$\mathbb{X} : (a, b) \times (0, 2\pi) \longrightarrow \mathbb{R}^3 : (t, \theta) \mapsto (x(t) cos\, \theta, x(t) sen\, \theta, z(t))$$

## IV. FOCAL SETS

The concept of *focal set*, which will be used throughout the paper, is required to characterize Canal surfaces. In this section we recall it and point out its main properties.

**Definition 4.1.** *A simple surface* $\mathrm{X} : U \longrightarrow \mathrm{R}^3$ *is called a* main chart *if the parametric curves* $u \mapsto \mathrm{X}(u, v)$ *and* $v \mapsto \mathrm{X}(u, v)$ *are lines of curvature.*

Let $M$ be a regular surface in $\mathrm{R}^3$. Let $W$ $M$ be a regular patch and $N$ a differentiable unit normal $N$. Let $\kappa_1$ and $\kappa_2$ be the principal curvatures of $M$ with respect to $N$ (we supp⊂ose them ordered so that $\kappa_1$ $\kappa_2$ on $W$). The reciprocalsof these curvatures are said to be the *principal radii of curvature* of $M$.

For each point $q$ in $M$, let us denote by $l_q$ the line normal to $M$ at $q$.≥Then, the surface normal $N(q)$ is a vector in $l_q$ and each normal section is a plane curve $\mathrm{C}$ in a plane $\Pi$ containing $l_q$. Note that the center of curvature of $\mathrm{C}$ lies on $l_q$ because $N(q)$ is perpendicular to $\mathrm{C}$ at $q$, Therefore, the centers of curvature at $q$ fill out a connected subset $F$ of $l_q$, which is called the *focal interval* of $M$ at $q$. The extremities of $F$ are called the *focal points* of $M$ at $q$.

Note that the focal interval $F$ reduces to a point if $q$ is itself an *umbilical* point of $M$,, that is to say, a point in which both in which the two principal curvatures coincide. Otherwise, $F$ it is a line segment, or the complement of a line seg- ment. Moreover, the focal points coincide if and only if $q$ is umbilical. When the Gaussian curvature of $M$ vanishes at $q$, then at least one focal point is at infinity.

The next figure 4.1 shows the behavior at an elliptic and a hyperbolic point.

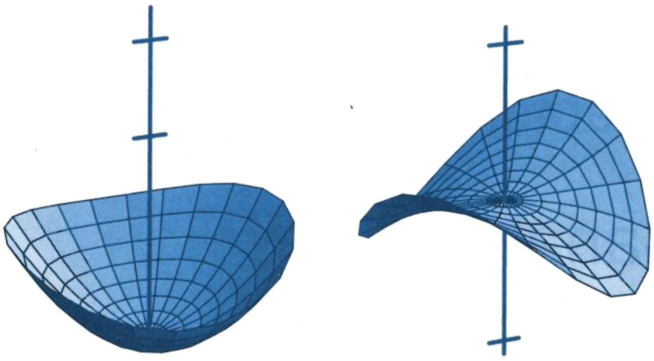

**Figure 4.1:** Focal interval in the vertices of an elliptic paraboloid and of an hyperbolic one

From here on, we suppose that both $\kappa_1$ y $\kappa_2$ are differentiable on $W$, and for convenience we take $W = M$. The set $FS(M) = \{p \in \mathbb{R}^3 \mid p \text{ is a focal point of some } q \in M\}$ is called the *focal set* of the regular surface $M \subset \mathbb{R}^3$.

When $M \subset \mathbb{R}^3$ is orientable, connected and with no umbilical points, its focal set has two components, one corresponding to each principal curvature. In these surfaces, there are three possibilities

Case 1: Each component of $FS(M)$ is a surface.

Case 2: One component of $FS(M)$ is a curve and the other a surface.

Case 3: Each component of $FS(M)$ is a curve.

Since Case 1 is the most general and thus the most studied, we will only deal in this paper with the other two cases, which are the Canal Surfaces and the Dupin cyclides. Both cases will be studied in the next sections.

## V. CANAL SURFACES

In this section we show the main properties of Canal surfaces, as well as their different types. New results on them are introduced and proved here.

### 5.1 Results on Canal surfaces

A 1-parameter family of surfaces can be described by a differentiable function of the type $F(x, y, z, t) = 0$, where $t$ is a parameter. When $t$ can be eliminated from the equations $F(x, y, z, t) = 0$, and $\dfrac{\partial F(x, y, z, t)}{\partial t} = 0$, we obtain the envelope, as a described implicity surface.

**Definition 5.1.** *A* Canal surface *is the envelope of a 1-parameter family $t \mapsto S^2(t)$ of spheres in $\mathbb{R}^3$. The curve which the centers of the spheres form is called the* center curve *of the Canal surface and the function $r$ such that $r(t)$ is the radius of the sphere $S^2(t)$ is called* radius *of the Canal surface.*

The following result, which is illustrated in Figure 5.1, is proved in [7].

Let $t \mapsto S^2(t)$ be the 1-parameter family of spheres defining a Canal surface $M$. Then for each $t$ the intersection $S^2(t) \cap M$ is a circle and a principal curve on $M$.

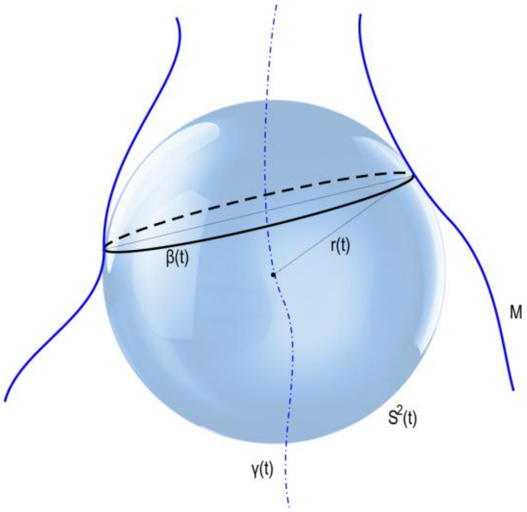


**Figure 5.1**: Construction of a Canal surface $M$, with a line of curvature $\beta(t)$ in $M$

Canal surfaces are characterized according to (Theorem 20.12 of [7])

**Theorem 5.2.** *Let $M$ $\mathrm{R}^3$ be a regular surface with no umbilical points. The following assertions are equivalent*

1. *$M$ is a Canal surface.* $\subset$
2. *One of the systems of principal curves of $M$ consists of circles.*

3. *One of the components of $FS(M)$ is a curve.*

An immediate consequence of this result is the following

**Proposition 5.3.** *Let $\alpha : (a,b) \to \mathbb{R}^3$ be a unit-speed curve with curvature different from zero. Let us suppose that $\alpha$ is the center curve of a Canal surface Then, this Canal surface can be parametrized by the mapping*

$$\mathbb{X}(t,\theta) = \alpha(t) + r(t)\left(-\mathbf{t}r'(t) + \sqrt{1 - r'(t)^2}(-\cos\theta\mathbf{n} + sen\ \theta\mathbf{b})\right),$$

*where* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ *denote the tangent, normal and binormal of* $\alpha$.

### 5.2 Examples of Canal surfaces

We show next three notable types of Canal surfaces.

#### 5.2.1 Tubular surfaces (colloquially tubes)

The set of points located at a distance $r$ from the curve $\alpha$ is called a *tubular surface* of radius $r$ around $\alpha$. As the normal and binormal are perpendicular to $\alpha$, the circle $\theta \mapsto r(-\cos\theta\mathbf{n}(t) + sen\ \theta\mathbf{b}(t))$ is perpendicular to $\alpha$ at $\alpha(t)$. As this circle moves along $\alpha$ it traces out a surface about $\alpha$, which will be the tube about $\alpha$, provided $r$ is not too large.

Note that a Canal surface is a tube when the radius function $r(t)$ is constant.

We can parametrize this surface by $\mathbb{X}(t,\theta) = \alpha(t) + r(-\cos\theta\mathbf{n}(t) + sen\ \theta\mathbf{b}(t))$, where $a \leq t \leq b$ and $0 \leq \theta \leq 2\pi$.

An interesting property of a tube about a curve $\alpha$ in $\mathbb{R}^3$ is the following: its volume depends only on the length of $\alpha$ and radius of the tube. Particularly, this volume depends neither the curvature nor torsion of $\alpha$. Thus, for example, tubes of the same radius about a circle and a helix of the same length will have the same volume. In [8] can be checked the proofs of these facts and the study of tubes in higher dimensions.

Tubes can be characterized among all Canal surfaces according to the follow-ing result

**Theorem 5.4.** *Let M be a Canal surface. The following conditions are equivalent*
1. *M is a tube parametrized as in Prop. 5.3.*
2. *The radius of M is constant.*
3. *The radius vector of each sphere in the family defining the Canal surface M meets the center curve orthogonally.*

*Proof.* By comparing (5.3) with the previously indicated parametrization $\mathbb{X}(t,\theta)$, we have that *1* and *2* are equivalent. The radius of each sphere is $\mathbb{X}(t,\theta) - \alpha(t)$. Then, by replacing the radius with $r = cte$ in the expression

$$\mathbb{X}(t,\theta) - \alpha(t) = -rr'\mathrm{t} - r\sqrt{1-r'^2}\cos\theta\mathbf{n} + r\sqrt{1-r'^2}sen\,\theta\mathbf{b}, \qquad (1)$$

we deduce that *2* and *3* are equivalent. It finishes the proof. □

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### 5.2.2 Surfaces of revolution

Surfaces of revolution are the most easily recognized class of surfaces. They are obtained as the result of rotating a curve $\alpha(t) = (x(t), z(t))$ about the z-axis provided $x(t) > 0$ is assumed to assure that $\alpha$ does not cross the axis of revolution. Then, it is well-known that the standard parametrization of the surface of revolution obtained is $\mathbb{X}(t, \theta) = (x(t)\cos\theta, x(t)sen\,\theta, z(t))$.
Next, we show some results of these surfaces. For the proofs of them, see [7].
The vast majority of surfaces of revolution are Canal surfaces. The following result characterizes these surfaces in the set of Canal surfaces.
**Theorem 5.5.** *The center curve of a Canal surface M is a straight line if and only if M is a surface of revolution for which no normal line to the surface is parallel to the axis of revolution. Furthermore, the parametrization of a Canal surface reduces to the standard parametrization given for a surface of revolution.*
The following assert gives a close link between evolutes of plane curves and focal sets of surfaces of revolution
**Theorem 5.6.** *Let $\alpha(t) = (x(t), z(t))$ be a plane curve that is neither a straight line nor part of a circle. Then, one of the components of the focal set of a surface of revolution M generated by $\alpha$ is the surface of revolution generated by the evoluteof $\alpha$.*

### 5.2.3 Dupin cyclides

The cyclides were discovered by Charles Dupin (1784-1873) in 1803, when he was an undergraduate and was studying the works by Gaspard Monge. Dupin called *cyclides* to those surfaces whose curvature lines are all circles. Many math- ematicians have analyzed these surfaces, giving new properties of them, as for example Darboux or Casey. However, it was in 1868, with the works by James Maxwell, when the interest in the cyclides was reborn.

In the first place, let us see which was the definition that Dupin gave in his book *Applications de Geometrie*, published in Paris in 1822.

**Definition 5.7.** *A cyclide is an envelope of spheres tangent to three given spheres.*
In any case, these surfaces can be also dealt with by using focal sets. To do this, we start from the previous definition and show that these surfaces can be seen as the envelope of spheres whose centers move along ellipses, hyperbolas orparabolas (as Maxwell defined them in 1868).

Indeed, let us consider the envelope as in the previous definition. If we fix three spheres of this envelope and reapplied the definition, we obtain a second envelope. All spheres of the second envelope are tangent to the fixed spheres of the first one. Since the election of fixed spheres from the first envelope is arbitrary, all spheres of the second envelope are tangent to all spheres of the first. Both envelopes obtained are Canal surfaces and, furthermore, they are complements of each other in the sense that the space swept by the spheres of the first envelope is the outside of the space swept by the spheres of the second envelope, and vice versa. Thus, they share a common surface which is the definition, a cyclide.
The curvature lines of each Canal surface form the curvature lines of the cy- clide. Hence, every cyclide can be thought of having a pair of Canal surfaces associated with it. By definition, the surface normals

of these Canal surfaces pass through two fixed curves: the curvature lines, which are circles, and the central curve of the Canal surface. It means that the definition given by Maxwell is ob- tained: the cyclide is a surface in which all its normal lines cut two fixed curves, which are ellipses, parabolas and hyperbolas.

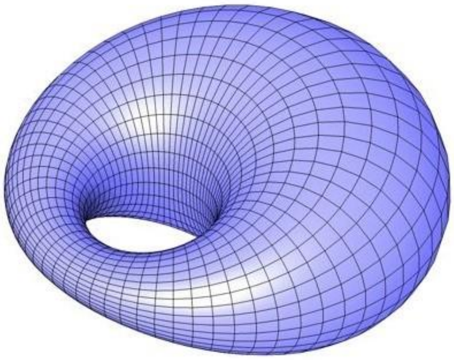

**Figure 5.2**: Ring cyclide

Now, focal sets consisting of two curves can be characterized. As it was said before, they are the Dupin cyclides. A proof of this last assert can be found in [7].

**Theorem 5.8.** *Let M be a surface for which the focal set consists of two curves. Then each curve is a conic section (an ellipse, hyperbola, parabola or straight line), and the planes of each component are perpendicular to one another.*

Starting from both the previous definition and the concept of focal sets, Dupin cyclides can be constructed. We show next two types of them: Elliptic-Hyperbolic and Parabolic cyclides.

- **Elliptic-Hyperbolic cyclides**

Let $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ be the ellipse in the xy-plane and the hyperbola $\frac{x^2}{c^2} - \frac{z^2}{a^2 - c^2} = 1$ in the xz-plane and we assume $a > c > 0$.

It is easy to see that the distance between a point $P = (x, y, 0)$ on the ellipse and a point $Q = (x', 0, z')$ on the hyperbola is given by the formula

$$d(P, Q) = \left| \frac{c}{a} x - \frac{a}{c} x' \right|.$$

Next, we parametrize both curves by $\alpha(u) = (a \cos u, \sqrt{a^2 - c^2}\, sen\ u, 0)$, and $\beta(v) = (c \sec v, 0, \sqrt{a^2 - c^2} \tan v)$,, respectively, being $\alpha$ the ellipse and $\beta$ the hyperbola, and we restrict to the branch of the hyperbola for which $-\pi/2 < v < \pi/2$.

**Lemma 5.9.** *Let $S_1(u)$ be a sphere with radius $\rho_1(u)$ centered at $\alpha(u)$, and let $S_2(v)$ be a sphere with radius $\rho_2(v)$ centered at $\beta(v)$. Suppose that $S_1(u)$ and $S_2(v)$ are tangent to each other. Then there exists $k$ such that*

$$\rho_1(u) = -c \cos u + k \quad \textit{and} \quad \rho_2(v) = a \sec v - k.$$

*Proof.* By hypothesis, the two spheres have a unique point in common, say $A = (x'', y'', z'')$. Then, $d(P, A) = \rho_1(u)$ *and* $d(Q, A) = \rho_2(v)$.

Since the spheres are tangent at $A$, we have $\rho_1(u) + \rho_2(v) = d(P, Q) = \left|\frac{c}{a} x - \frac{a}{c} x'\right| = |c \cos u - a \sec v| = a \sec v - c \cos u$.

Thus, $\rho_1(u) + c \cos u = -\rho_2(v) + a \sec v$. Since the left-hand side of the previous expression depends only on $u$, while the right-hand side only on $v$, we let $k$ be the common value $\rho_1(u) + c \cos u = -\rho_2(v) + a \sec v = k$, and we obtain the result. □

Let us now prove the converse of this Lemma.

**Lemma 5.10.** *Let $S_1(u)$ be a sphere with radius $\rho_1(u)$ centered at $\alpha(u)$, and let $S_2(v)$ be a sphere with radius $\rho_2(v)$ centered at $\beta(v)$. Suppose that $\rho_1(u)$ and $\rho_2(v)$ are given as in the previous Lemma, for some $k$. Let $A = (x'', y'', z'')$ be the point on the line connecting the points $\alpha(u)$ and $\beta(v)$ at a distance $\rho_1(u)$ from $\alpha(u)$. Then, the distance between the points $A$ and $\beta(v)$ is $\rho_2(v)$. Furthermore, $S_1(u)$ y $S_2(v)$ are mutually tangent at $A$.*

*Proof.* The distance from $\alpha(u)$ and $\beta(v)$ is $d((x, y, 0), (x', 0, z')) = \left|\frac{c}{a}x - \frac{a}{c}x'\right| = |c\cos u - a\sec v| = a\sec v - c\cos u = \rho_1(u) + \rho_2(v)$.

This can only happen if $S_1(u)$ and $S_2(v)$ are tangent at $A = (x'', y'', z'')$. $\square$

As $S_1(u)$ moves along the ellipse $\alpha(u)$ it traces out a surface, the envelope of the spheres. This envelope must coincide with the envelope formed by $S_2(v)$ as it moves along a branch of the hyperbola $\beta(v)$. The resulting surface is called a *Dupin cyclide of elliptic-hyperbolic type*. It depends on three parameters $a$, $c$ and $k$.

In order to parametrize the cyclide, we define $\mathbb{X}(u, v) = \alpha(u) + \rho_1(u)P(u, v) = \beta(v) - \rho_2(v)P(u, v)$, where $P(u, v)$ is a unit vector. We have $0 = \alpha(u) - \beta(v) + (\rho_1(u) + \rho_2(v))P(u, v)$, whence

$$\mathbb{X}(u, v) = \alpha(u) - \rho_1(u)\left(\frac{\alpha(u) - \beta(v)}{\rho_1(u) + \rho_2(v)}\right) = \frac{\rho_2(v)\alpha(u) + \rho_1(u)\beta(v)}{\rho_1(u) + \rho_2(v)}.$$

Substituting in the previous expressions and by denoting $d = k - c\cos u$ and $e = a - k\cos v$, we obtain the parametrization of the cyclide

$$\mathbb{X}(u, v) = \left(\frac{cd + ae\cos u}{a - c\cos u\cos v}, \frac{e\sqrt{a^2 - c^2}\,\mathrm{sen}\ u}{a - c\cos u\cos v}, \frac{d\sqrt{a^2 - c^2}\,\mathrm{sen}\ v}{a - c\cos u\cos v}\right).$$

There are three classes of the cyclide of elliptic-hyperbolic type

- The “Ring cyclide” with no self-intersections.
- The “Spindle cyclide” with 1 self-intersection.
- The “Horn cyclide” with 2 self-intersections.

Figure 5.2.3 shows a ring cyclide. A spindle cyclide and a horn cyclide can be checked, respectively, in the following figures 5.3 and 5.4

- **Parabolic cyclides**

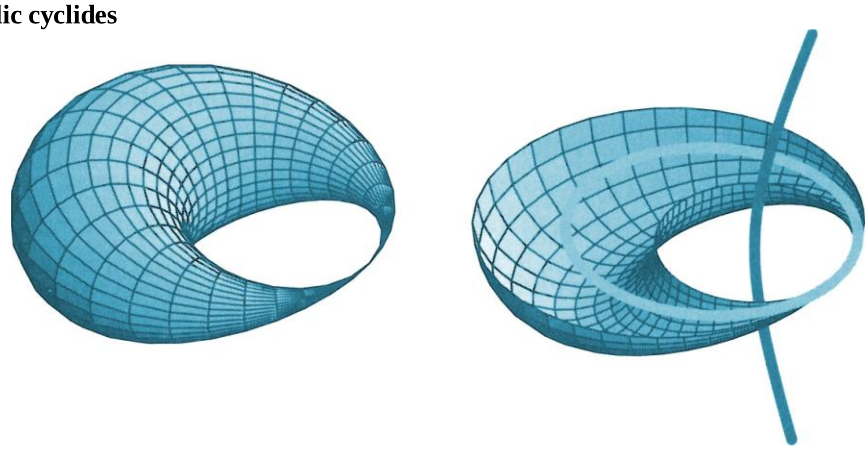

Figure 5.3: Spindle cyclide and its focal set

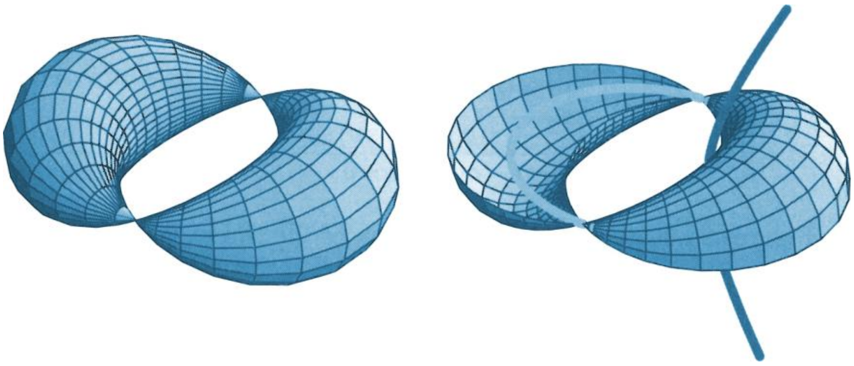

Figure 5.4: Horn cyclide and its focal set

There is another type of cyclide whose focal set consists of two parabolas, $z = -\frac{x^2}{8a} + a$, in the xz-plane and $z = \frac{y^2}{8a} - a$, in the yz-plane, respectively parametrized by

$$\alpha(u) = \left(u,\ 0,\ -\frac{u^2}{8a} + a\right),\ \beta(v) = \left(0,\ -v,\ \frac{v^2}{8a} - a\right).$$

The following result and its proof are similar to the ones of Lemma 5.9.

**Lemma 5.11.** *Let $S_1(u)$ be a sphere with radius $\rho_1(u)$ centered at $\alpha(u)$, and let $S_2(v)$ be a sphere with radius $\rho_2(v)$ centered at $\beta(v)$. Suppose that $S_1(u)$ and $S_2(v)$ are tangent to each other. Then there exists $k$ such that*

$$\rho_1(u) = \frac{u^2}{8a} + a + k \quad \textit{and} \quad \rho_2(v) = \frac{v^2}{8a} + a - k.$$

We can parametrize the parabolic cyclide (see Figure 5.5) as follows: $\mathbb{X}(u, v) =$

$$\left(\frac{u(8a^2 + k + v^2)}{e}, \frac{v(8a^2 - k + u^2)}{e}, \frac{16a^2k - 16a^2u^2 - ku^2 + 16a^2v^2 - kv^2}{8ae}\right),$$

where $e = 16a^2 + u^2 + v^2$.

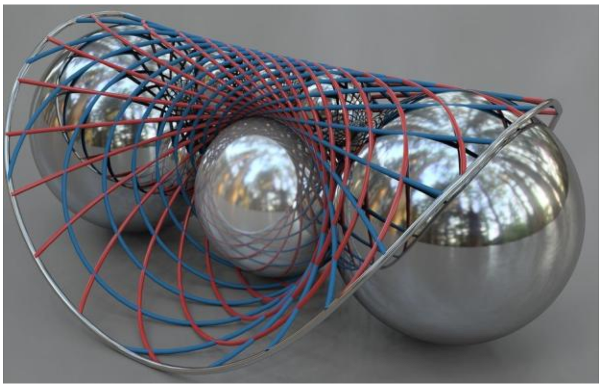

Figure 5.5: Parabolic Cyclide

## VI. Some applications of Canal surfaces

There exist lot of papers in the literature devoted to study blending surfaces, which play a relevant role within Canal surfaces. As it has been indicated in the Intro- duction, blending surfaces are obtained starting from a certain number of Canal surfaces, called primary surfaces, which generate one or more auxiliary surfaces which create a differentiable transition between the first ones. The final geomet- rical object obtained is the result of the unions of all of them in a unique piece. These transitions are operations necessary in the mechanical unions to make the objects differentiable. Many of the objects that we find in Engineering are el- ementary surfaces (cylinders, cones, spheres or planes, for instance), but in the case of blending surfaces it is necessary to work with free-form surfaces, that is to say, surfaces of non-elementary forms. In particular, the books [11] and [12] are outstanding references.

One of the most commonly used tools for 3D modeling today is the Blender program (so called for constructing the aforementioned blending surfaces). Its potential is enormous: in addition to modeling, it can be used to make special effects, develop video-games or create animations. Indeed, in January 1995, the Dutch animation studio "Neo Geo" developed Blender as an in-house application according to the works by Ton Roosendaal. Curiously, the name *Blender* was taken from a song by Yello, from the album Baby).

When Neo Geo was sold to another company, Ton Roosendaal and Frank van Beek founded "Not a Number Technologies" (NaN) in June 1998 with the aim of further developing Blender. Later, on July 18, 2002, Roosendaal started the "Free Blender" campaign, which was a crowdfunding precursor. Nowadays, Blender is a free software.

Spider-Man 2 was the first professional project that used Blender to create animations for the storyboard department. After, Blender was also used by NASA for publicly available 3D models.

Figure 6.1: The living room of a house designed with Blender: penultimate stage

In Architecture, the Blender program can also be used for several applications. For instance, in the design of interiors of a living place. In this way, before the construction of a house we are able to imagine the final result and can enter all the details we want. Each figure is a Blending Surface, and it is constructed using the Canal surfaces. In this way, the transitions between each part of the figure are differentiable, which gives an effect much more similar to reality. For example, Figures 6.1 and 6.2 show the penultimate and the last stages of the design of the living room of a house made by authors using this application.

Figure 6.2: The living room of a house designed with Blender: last stage

CAGD is also frequently used in Engineering. For instance, specific appli- cations of it have given rise to various specific design tools, in which this type of discipline is oriented to the elaboration of models restricted to a certain conceptual framework. Thus, CAGD applications have been developed aimed at the elabora- tion of models in the Aerospace, Automotive or Naval Industry. Within the latter, in this century, Computer Aided Graphic Design of Ships (CASGD) has gained special importance. Although in the different Teaching systems CAGD techniques tend to be approached in a generalist way without descending to specific applica- tions of the degree in which they are included, Suffo showed in [15] the adaptation of a CASGD methodology based on NURBS surfaces (Non-Uniform Rational B- splines) for the teaching of this type of discipline in Naval Technical Engineering.

With respect to NURBS, there is recent research to solve the limitations of the NURBS standard, which are basically of two types. In the first place, it does not include transcendent curves and surfaces, some of them as common in Engi- neering as the propeller, the catenary, or the spirals. A second limitation comes from the fact that the maximum degree that CAD programs support is limited for efficiency reasons. Consequently, certain high-grade entities, such as surfaces resulting from global deformation, cannot be included in the NURBS standard either. Further information about the use of curves and surfaces in CAGD can b checked in [4, 10].

Finally, another use of the Blender is, as we mentioned before, the modeling for the later animation of a figure. In this way, we can give *life* to any drawing or figure. It allows us to apply Blender in various branches of Biology, for instancein Anatomy.

For instance, if we wish to animate the figure of a doll playing rugby, in attack position, first, we build his whole body by using circumferences that will be the intersections of the spheres with the figure. One can also see the axes inside the body of the doll.

In the following stage, one can see the "bones", which are the aforementioned axes, where the centers of the spheres are located with which the object is going to be modeled. In this way, each part of the body is a channel surface, and the union of one with another is a differentiable transition, which makes the object to be totally well defined.

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