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On Double Laplace Transform and Double Sumudu Transform

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Abstract: *In this paper, double Sumudu transform method is introduced used to solve the one dimensional heat equation and the results are compared with the results of double Laplace transform*
Keyword: *Double Laplace transforms, Double Sumudu transform, heat equation*

I. INTRODUCTION

The heat equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time. Head is the energy transferred from one point to another. heat flows from the point of higher temperature to one of lower temperature. partial differential equation that governs the heat flow in a rod.
The PDE can be formally shown to satisfy

$U_t = kU_{xx}, \quad 0 < x < L, \; t > 0 \quad (1)$

Where $U = U(x,t)$ represents the temperature of the rod at the position x at time t, and k is the thermal diffusivity of the material that measures the rod ability to heat conduction.
In recent years, many researches have paid attention to find the solution of partial differential equations by using various methods. Among these are the double Laplace transform, and double Sumudu transform , there are various ways have been proposed recently to deal with these partial differential equations, one of these combination is Sumudu transform method . The Sumudu transform a kind of modified Laplace's transform.

II. LAPLACE TRANSFORMS

Laplace transforms of the function $f(t)$ is denoted by $L[f(t)]$ and defined as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt. \qquad (2)$$

We assume that this integral exists.

Let $f(x,t)$ be a function that can be express as convergent infinite series and let $(x,t) \in R_+^n$ then the double Laplace transform of a function of $f(x,t)$ in positive quadrant of xt - plane is given by.

$$L_x L_t[f(x,t):(p,s)] = \int_0^\infty \int_0^\infty f(x,t) e^{-(px+st)} dx\,dt = F(p,s) \qquad (3)$$

Where $x, t \ge 0$, and p, s are transform variables for x and t respectively whenever the improper integral is convergent.
If $f(x,t)$ is continuous function have second partial derivative, then double Laplace transform of partials derivative of the first and second as follow:-

- Double Laplace transform for first partial derivative with respect to t is

$$\left[\frac{\partial f(x,t)}{\partial t};(p,s)\right] = \int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-(px+st)} dx\,dt = sF(p,s) - F(p,0) \qquad (4)$$

- Double Laplace transform for first partial derivative with respect to x is

$$L_x\left[\frac{\partial f(x,t)}{\partial x};(p,s)\right] = \int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-(px+st)} dx\,dt = pF(p,s) - F(0,s) \qquad (5)$$

- Double Laplace transform for second partial derivative with respect to t is

$$L_t L_t\left[\frac{\partial^2 f(x,t)}{\partial t^2};(p,s)\right]=\int_0^\infty\int_0^\infty \frac{\partial^2 f(x,t)}{\partial^2 t}e^{-(px+st)}dx\,dt=s^2F(p,s)-sF(p,0)-\frac{\partial F(x,0)}{\partial t} \quad (6)$$

- Double Laplace transform for second partial derivative with respect to x is

$$L_x L_x\left[\frac{\partial^2 f(x,t)}{\partial^2 x};(p,s)\right]=\int_0^\infty\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x^2}e^{-(px+st)}dx\,dt=p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}$$

(7)

III. SUMUDU TRANSFORMATIONS

Sumudu transform of a function $f(t)$ is defined for all real numbers $t>0$ as the function $S(u)$, given by:

$$S[f(t)]=M(s)=\frac{1}{v}\lim_{j\to\infty}\int_0^j e^{-\frac{t}{v}}f(t)dt. \quad (8)$$

Let $f(x,t)$ be a function that can be express as convergent infinite series and let $(x,t)\in R_+^n$ then the double Sumudu transform of the function of $f(x,t)$ in positive quadrant of xt - Plane is given by

$$S_2[f(x,t):(u,v)]=\frac{1}{u}\frac{1}{v}\int_0^\infty\int_0^\infty f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx\,dt \quad (9)$$

Where $x,t\geq 0$, and u, v are transform variables for x and t respectively whenever the improper integral is convergent.

If $f(x,t)$ is continuous function have second partial derivative, then Sumudu transform of partials derivative of the first and second as follow:

- Double Sumudu transform for first partial derivative with respect to t is

$$S_2\left[\frac{\partial f(x,t)}{\partial t};(\mathrm{u},\mathrm{v})\right]=\frac{1}{u}\frac{1}{v}\lim_{j\to\infty}\int_0^j\int_0^j\frac{\partial f(x,t)}{\partial t}e^{-\frac{x}{u}}.\,e^{-\frac{t}{v}}dx\,dt=\frac{1}{v}[M(u,v)-f(u,0)] \quad (10)$$

- Double Sumudu transform for first partial derivative with respect to x is

$$S_2\left[\frac{\partial f(x,t)}{\partial x};(\mathrm{u},\mathrm{v})\right]=\frac{1}{v}\int_0^\infty\left[\frac{1}{u}\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-\frac{x}{u}}dx\right].\,e^{-\frac{t}{v}}dt=\frac{1}{u}[M(u,v)-M(0,v)]$$

(11)

- Double Sumudu transform for second partial derivative with respect to t

$$S_2\left[\frac{\partial^2 f(x,t)}{\partial t^2};(\mathrm{u},\mathrm{v})\right]=\frac{1}{u}\frac{1}{v}\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial t^2}e^{-\frac{x}{u}}.\,e^{-\frac{t}{v}}dx\,dt=\frac{1}{v^2}M(u,v)-\frac{1}{v^2}M(u,0)-\frac{1}{v}\frac{\partial M(u,0)}{\partial t}$$

(12)

- Double Sumudu transform for second partial derivative with respect to x is :-

$$S_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(\mathrm{u},\mathrm{v})\right]=\frac{1}{u}\frac{1}{v}\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-\frac{x}{u}}.\,e^{-\frac{t}{v}}dx\,dt=\frac{1}{u^2}M(u,v)-\frac{1}{u^2}f(0,v)-\frac{1}{u}\frac{\partial M(0,v)}{\partial x}(13)$$

IV. APPLICATIONS

In this section, we assume that the inverse double Sumudu transform is exists. We apply the inverse double Sumudu transform to find the solution of the heat equation in one dimension with initial and boundary conditions.

Example (1)

Solving the heat equation

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2},\quad t>0,\ (14)$$

With conditions

$$U(0,t)=0, \qquad U(x,0)=\sin x, \quad \frac{\partial U(0,t)}{\partial x}=e^{-t} \quad (15)$$

By taking the double Laplace transform of equation (14), we get

$$sF(p,s)-F(p,0)=p^{2}F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x} \quad (16)$$

The single Laplace transform of initial conditions gives

$$F(0,s)=0, \quad F(p,0)=\frac{1}{p^{2}+1}, \quad \frac{\partial F(0,s)}{\partial x}=\frac{1}{s+1} \quad (17)$$

By substituting (17) into equation (16), we get

$$sF(p,s)-\frac{1}{p^{2}+1}=p^{2}F(p,s)-0-\frac{1}{s+1}$$

$$\left(s-p^{2}\right)F(p,s)=\frac{\left(s-p^{2}\right)}{\left(p^{2}+1\right)\left(s+1\right)}$$

$$F(p,s)=\left(\frac{1}{p^{2}+1}\right)\left(\frac{1}{s+1}\right) \quad (18)$$

Applying inverse double Laplace transform of equation (18) gives thesolution of heat equation (5.3) in the form

$$U(x,t)=\mathrm{e}^{-t}\sin x\,. \quad (19)$$

By taking the double Sumudu transform of equation (14), we get:-

$$\frac{1}{v}\left[M(u,v)-M(u,0)\right]=\frac{1}{u^{2}}M(u,v)-\frac{1}{u^{2}}M(0,v)-\frac{1}{u}\frac{\partial M(0,v)}{\partial x} \quad (20)$$

The single Sumudu transformation for the conditions gives

$$M(0,v)=0, \qquad M(u,0)=\frac{u}{1+u^{2}} \qquad \frac{\partial M(0,v)}{\partial x}=\frac{1}{1+v} \quad (21)$$

Substitute (21) in (20) we get

$$\frac{u^{2}-v}{u^{2}v}M(u,v)=\frac{1}{v}\frac{u}{1+u^{2}}-\frac{1}{u}\frac{1}{1+v}$$

$$M(u,v)=\frac{u}{1+u^{2}}\frac{1}{1+v} \quad (22)$$

Applying inverse double Sumudu transform of equation (22) gives the solution of heat equation (14) in the form

$$U(x,t)=\sin x\ e^{-t}.(23)$$

Example (2)

Solving the heat equation

$$\frac{\partial U}{\partial t}=\frac{\partial^{2}U}{\partial x^{2}}+\sin x, \quad t>0 \quad (24)$$

With conditions

$$U(0,t)=e^{-t},\quad U(x,0)=\cos x,\quad \frac{\partial U(0,t)}{\partial x}=1-e^{-t} \qquad (25)$$

By taking the double Laplace transform to Eq (24) we get

$$sF(p,s)-F(p,0)=p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}+\frac{1}{sp^2+s} \qquad (26)$$

The single Laplace transform of initial conditions gives

$$F(p,0)=\frac{p}{p^2+1},\quad \frac{\partial F(0,s)}{\partial x}=\frac{1}{s^2+s},\qquad F(0,s)=\frac{1}{s+1} \qquad (27)$$

By substituting (27) into equation(26), we get

$$\left(s-p^2\right)F(p,s)-=\frac{p}{p^2+1}-\frac{p}{s+1}+\frac{1}{sp^2+s}-\frac{1}{s^2+s}$$

$$\left(s-p^2\right)F(p,s)-=\frac{p(s-p^2)}{\left(p^2+1\right)\left(s+1\right)}+\frac{s(s-p^2)}{\left(sp^2+s\right)\left(s^2+s\right)}$$

$$F(p,s)=\frac{p}{\left(p^2+1\right)\left(s+1\right)}+\frac{1}{s}\frac{1}{p^2+1}\frac{s}{s^2+s}$$

$$F(p,s)=\frac{p}{p^2+1}\frac{1}{s+1}+\frac{1}{p^2+1}\left(\frac{1}{s}-\frac{1}{s+1}\right) \qquad (28)$$

Applying inverse double Laplace transform of equation (28) gives the solution of heat equation (24) in the form:-

$$U(x,t)=\cos x\ \ e^{-t}+\sin x\,(1-e^{-t}) \qquad (29)$$

By taking the double Sumudu transform of equation (24) we get

$$\frac{1}{v}\left[M(u,v)-M(u,0)\right]=\frac{1}{u^2}M(u,v)-\frac{1}{u^2}M(0,v)-\frac{1}{u}\frac{\partial M(0,v)}{\partial x}+\frac{u}{1+u^2}$$

$$\frac{u^2-v}{u^2v}M(u,v)=\frac{1}{v}M(u,0)-\frac{1}{u^2}M(0,v)-\frac{1}{u}\frac{\partial M(0,v)}{\partial x}+\frac{u}{1+u^2} \qquad (30)$$

The single Sumudu transformation for the conditions gives:-

$$M(0,v)=\frac{1}{1+v},\quad M(u,0)=\frac{1}{1+u^2},\quad \frac{\partial M(0,v)}{\partial x}=1-\frac{1}{1+v} \qquad (31)$$

Substitute (30) in (29) we get

$$\frac{u^2-v}{u^2v}M(u,v)=\frac{1}{v}\frac{1}{1+u^2}-\frac{1}{u^2}\frac{1}{1+v}-\frac{1}{u}\left(1-\frac{1}{1+v}\right)+\frac{u}{1+u^2}$$

$$\left(u^2-v\right)M(u,v)=\left(\frac{u^2}{1+u^2}-\frac{v}{1+v}\right)+\left(\frac{u^3v}{1+u^2}-\frac{uv^2}{1+v}\right)$$

$$M(u,v)=\frac{1}{1+u^2}\frac{1}{1+v}+\frac{u}{1+u^2}\left(1-\frac{1}{1+v}\right) \qquad (32)$$

Applying inverse double Sumudu transform of equation (36) gives the solution of heat equation (24) in the form

$$U(x,t)=\cos x\ \ e^{-t}+\sin x\,(1-e^{-t}) \qquad (33)$$

Example (3)

Solving the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-3u+3,\quad t>0, \qquad (34)$$

With conditions

$$u(0,t)=1, \qquad u(x,0)=1+\sin x, \qquad \frac{\partial u(0,t)}{\partial x}=e^{-4t} \qquad (35)$$

By taking the double Laplace transform to Eq (34) we get

$$sF(p,s)-F(p,0)=p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}-3F(p,s)+\frac{3}{ps} \qquad (36)$$

The single Laplace transform of initial conditions gives

$$F(p,0)=\frac{1}{p}+\frac{1}{p^2+1}, \qquad F(0,s)=\frac{1}{s}, \qquad \frac{\partial F(0,s)}{\partial x}=\frac{1}{s+4} \quad (37)$$

By substituting (37) into equation (36), we get

$$sF(p,s)-\left(\frac{1}{p}+\frac{1}{p^2+1}\right)=p^2F(p,s)-p\frac{1}{s}-\frac{1}{s+4}-3F(p,s)+\frac{3}{ps}$$

$$\left(s-p^2+3\right)F(p,s)=\frac{s+4-p^2-1}{\left(p^2+1\right)\left(s+4\right)}+\frac{s-p^2+3}{ps}$$

$$F(p,s)=\frac{1}{\left(p^2+1\right)\left(s+4\right)}+\frac{1}{ps} \quad \text{Or}$$

$$F(p,s)=\frac{1}{p^2+1}\frac{1}{s+4}+\frac{1}{ps} \qquad (38)$$

Applying inverse double Laplace transform of equation (38) gives the solution of heat equation (34) in the form

$$U(x,t)=\sin x \quad e^{-4t}+1. \qquad (39)$$

By taking the double Sumudu transform to Eq (34) we get

$$\frac{1}{v}\left[M(x,v)-M(u,0)\right]=\frac{1}{u^2}M(u,v)-\frac{1}{u^2}M(0,v)-\frac{1}{u}\frac{\partial M(0,v)}{\partial x}-3M(u,v)+3. \qquad (40)$$

The single Sumudu transform of initial conditions gives

$$M(u,0)=1+\frac{u}{1+u^2}, \qquad M(0,v)=1, \qquad \frac{\partial M(0,v)}{\partial x}=\frac{1}{1+4v}. \qquad (41)$$

By substituting (41) into equation (40), we get

$$\frac{1}{v}\left[M(x,v)-\left(1+\frac{u}{1+u^2}\right)\right]=\frac{1}{u^2}M(u,v)-\frac{1}{u^2}-\frac{1}{u}\frac{1}{1+4v}-3M(u,v)+3$$

$$\Rightarrow \left(u^2-v+3u^2v\right)M(u,v)=\frac{u^3}{1+u^2}-\frac{uv}{1+4v}+u^2-v+3u^2v$$

$$\left(u^2-v+3u^2v\right)M(u,v)=\frac{u^3+4u^3v-uv-u^3v}{\left(1+u^2\right)\left(1+4v\right)}+\left(u^2-v+3u^2v\right)$$

$$M(u,v)=\frac{u}{\left(1+u^2\right)\left(1+4v\right)}+1$$

$$M(u,v)=\frac{u}{1+u^2}.\frac{1}{1+4v}+1 \qquad (42)$$

Applying inverse double Sumudu transform of equation (42) gives the solution of heat equation (34) in the form

$$U(x,t)=\sin x \quad e^{-4t}+1. \qquad (43)$$

V. CONCLUSIONS

Double Sumudu transform is applied to obtain the solution ofheat equation of one dimensional, the result are compared with result of double Laplace transform. The heat equation in one dimensional under the boundary conditions, give similar results when we use the double Sumudu transform and double Laplace transform

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