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Solution of the Linear and Non- Linear Schrodinger Equation Using Homotopy Perturbation Method and Variational Iteration Method

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ABSTRACT: *This paper applies, homotopy perturbation method (HPM) and Variational Iteration Method (VIM) are employed to solve the linear and nonlinear Schrodinger equations. To illustrate the capability and reliability of the methods, some examples are provided. The results obtained using homotopy perturbation method (HPM) are compared with result Variational Iteration Method (VIM).*

Keywords: *Schrodinger equation,homotopy perturbation method, Variational Iteration Method.*

I. INTRODUCTION

The Schrodinger equation plays the role of Newton's laws and conservation of energy in classical mechanics. The Schrödinger equation does not give the trajectory of a particle, but rather the wave function of the quantum system, which carries information about the wave nature of the particle, which allows us to only discuss the probability of finding the particle in different regions of space at a given moment in time. The Schrodinger equation has two 'forms', one in which time explicitly appears, and so describes how the wave function of a particle will evolve in time. In general, the wave function behaves like a wave, and so the equation is often referred to as the time dependent Schrodinger wave equation. The other is the equation in which the time dependence has been 'removed' and hence is known as the time independent Schrodinger equation and is found to describe, amongst other things, what the allowed energies are of the particle. These are not two separate, independent equations the time independent equation can be derived readily from the time dependent equation.

In recent years, many researches have paid attention to find the solution of Schrodinger equation by using various methods. Among these are the a Spectral Method [1], Split step method [2], Adomian decomposition Method [11], the Wentzel–Kramers–Brillouin (WkB) Method[15], Nikiforod-Uvarov (NU) Method [21], Variational Iteration method have been studied by a numbers of authors [3, 4, 5, 6, 10, 12],and homotopy perturbation method(HPM) [7, 8, 11, 13, 14, 16, 17, 18], various ways have been proposed recently to deal with these Schrodinger equation.

i. The linear Schrodinger equation of the form

$$\frac{\partial \Psi(x,t)}{\partial t} = i\,\frac{\partial^2 \Psi(x,t)}{\partial x^2}, \quad i^2 = -1, \ t > 0 \quad (1)$$

$$\Psi(x,0) = f(x)$$

where $f(x)$ is continuous and square integrable,

The linear Schrodinger equation with initial value problem for a free particle with mass m is given by standard form

$$\frac{\partial \Psi(x,t)}{\partial t} = i\,\frac{\partial^2 \Psi(x,t)}{\partial x^2}, \quad x \in \mathbf{R},\ t > 0 \quad (2)$$

$$\Psi(x,0) = \alpha e^{ikx}$$

Where α and k are constants. It is to be noted that Schrodinger equation (2) discusses the time evolution of a free particle. Moreover, the function $\Psi(x,t)$ is complex and equation (2) is a first order differential equation in t. The Schrodinger equation (2) is usually handled by using the spectral transform among other methods.

ii. The nonlinear Schrodinger equation of the form

$$i\frac{\partial\Psi(x,t)}{\partial t}+\frac{\partial^2\Psi(x,t)}{\partial x^2}+\gamma\left|\Psi\right|^2\Psi=0 \quad (3)$$

$$\Psi(x,0)=g(x)$$

With $\Psi=\Psi(x,0)$ is differentiable function, $g(x)$ is the initial value and γ is a constant.The nonlinear Schrodinger equation (NLS) with initial value problem defined by its standard form

$$i\frac{\partial\Psi(x,t)}{\partial t}+\frac{\partial^2\Psi(x,t)}{\partial x^2}+\gamma\left|\Psi\right|^2\Psi, \qquad (4)$$

$$\Psi(x,0)=e^{ikx}$$

Where γ a constants and function $\Psi(x,t)$ is complex.

The nonlinear Schrodinger equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena.

II. HOMOTOPY PERTURBATION METHOD

To illustrate the basic idea of homotopy perturbation method (HBM), we consider thefollowing non-linear differential equation:

$$A(\Psi)-f(\mathrm{r})=0,\quad \mathrm{r}\in\Omega \qquad (5)$$

With the following boundary conditions:

$$B\left(\Psi,\frac{\partial\Psi}{\partial n}\right)=0,\quad \mathrm{r}\in\Gamma \qquad (6)$$

Where A is a general differential operator, B is a boundary operator, $f(\mathrm{r})$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be decomposed into a linear and a non-linear, designated as L and N respectively. The equation (5) can be written as the following form.

$$L(\Psi)+N(\Psi)-f(r)=0 \qquad (7)$$

Using homotopy perturbation technique, we construct a homotopy $\psi(r,p):\Omega\times[0,1]\rightarrow \boldsymbol{R}$ which satisfies

$$H(\psi,p)=(1-p)\left[L(\psi)-L(\Psi_0)\right]+p\left[A(\psi)-f(r)\right]=0 \qquad (8)$$

Where $p\in(0,1)$ is an embedding parameter, Ψ_0 is an initial approximation solution of (5), which satisfies the boundary, form equation (8) we obtain

$$H(\psi,0)=L(\psi)-L(\Psi_0)=0 \qquad (9)$$

$$H(\psi,1)=A(\psi)-f(r)=0 \qquad (10)$$

Changing the process of p from zero to unity, a change $\psi(r,p)$ from $\Psi_0(r)$ to $\Psi(r)$.

In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solutions of equation (8) can be written as a power series in p as the following

$$\psi=\psi_0+p\psi_1+p^2\psi_2+p^3\psi_3+\cdots \qquad (11)$$

Setting $p=1$ results in the approximate of equation (11), can be obtained

$$\Psi=\lim_{p\to1}\psi=\psi_0+\psi_1+\psi_2+\psi_3+\cdots \qquad (12)$$

III. VARIATINAL ITERATION METHOD

The variational iteration method (VIM) established by Ji-Huan He [1] is thoroughly used by mathematicians to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous as well. It was shown that this method is effective and reliable for analytic and numerical purposes. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM does not require specific treatments for nonlinear problems as in Adomian method, perturbationtechniques, etc. In what follows, we present the main steps of the method. Consider the differential equation

$$L(\Psi(x,t)) + N(\Psi(x,t)) = g(x,t) \quad (13)$$

Where L and N are linear and nonlinear operator and $g(x,t)$ is sourceinhomogeneous term. The variational iteration method, we can construct a correction function for question (13) in the form

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \int \lambda(\zeta)\left(L\Psi(x,\zeta) + N\tilde{\Psi}(x,\zeta) - g(x,\zeta)\right)d\zeta, \quad n \geq 0 \quad (14)$$

Where λ is a general Lagrangian multiplier, which can be identified optimally via the variation theory, the second term on the right is called the correction and $\tilde{\Psi}$ is a restricted variation which means $\delta\tilde{\Psi}=0$.

We first determine the Lagrange multiplier λ that will be identified optimally via integration b parts. The successive approximations $\Psi_{n+1}(x,t)$ of the solution of $\Psi(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective $\Psi_0(x,t)$ function. Consequently, the solution

$$\Psi(x,t) = \lim_{n\to\infty} \Psi_n(x,t) \quad (15)$$

IV. APPLICATIONS

In this section, we apply the homotopy perturbation method(HBM) and variational iteration method (VIM) to some linear and nonlinear partial differential equations.

Example.1

Use the Homotopy perturbation method to solve the linear Schrodinger equation:

$$\frac{\partial\Psi(x,t)}{\partial t} + i\frac{\partial^2\Psi(x,t)}{\partial x^2} = 0, \quad x \in \mathbf{R},\ t > 0 \quad (16)$$
$$\Psi(x,0) = \alpha e^{ikx}$$

Using HPM, we construct a homotopy in the following form

$$H(\psi,p) = (1-p)\left[\frac{\partial\psi}{\partial t} - \frac{\partial\Psi_0}{\partial t}\right] + p\left[\frac{\partial\psi}{\partial t} + i\frac{\partial^2\psi}{\partial x^2}\right] = 0 \quad (17)$$

We select $\Psi_0(x,t) = \alpha e^{ikx}$ as in initial approximation that satisfies the two conditions. Substituting equation (11) into equation (16) end equating the terms with identical powers of p, we drive

$$p^0: \begin{cases} \dfrac{\partial\psi_0}{\partial t} - \dfrac{\partial\Psi_0}{\partial t} = 0, \\ \psi_0(x,0) = \alpha e^{ikx}, \end{cases} \quad (18)$$

$$p^1: \begin{cases} \dfrac{\partial\psi_1}{\partial t} + \dfrac{\partial\Psi_0}{\partial t} + i\dfrac{\partial^2\psi_0}{\partial x^2} = 0, \\ \psi_1(x,0) = 0, \end{cases} \quad (19)$$

$$p^2: \begin{cases} \dfrac{\partial \psi_2}{\partial t} + i\dfrac{\partial \psi_1}{\partial x^2} = 0, \\ \psi_2(x,0) = 0, \end{cases} \quad (20)$$

$$p^3: \begin{cases} \dfrac{\partial \psi_3}{\partial t} + i\dfrac{\partial \psi_2}{\partial x^2} = 0, \\ \psi_3(x,0) = 0, \end{cases} \quad (21)$$

$$\vdots$$

Consider $\psi_0 = \Psi_0(x, t) = \alpha e^{ikx}$. Form equations (18), (19) ,(20) and (21), we have

$$\psi_1 = \frac{\alpha(k^2 it)}{1!} e^{ikx}$$

$$\psi_2 = \frac{\alpha(k^2 it)^2}{2!} e^{ikx}$$

$$\psi_3 = \frac{\alpha(k^2 it)^3}{3!} e^{ikx}$$

Therefore, the solution of equation (16) when $p \to 1$ we will be as follows:

$$\Psi(x, t) = \alpha\left(1 + \frac{(k^2 it)}{1!} + \frac{(k^2 it)^2}{2!} + \frac{(k^2 it)^3}{3!} + \cdots\right) e^{ikx} = \alpha e^{i\,k(x+kt)} \quad (22)$$

Variational Iteration Method:

By using variational iteration method for equation (16), we obtained

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \int_0^t \lambda(\zeta)\left(\frac{\partial \Psi_n(x,\zeta)}{\partial \zeta} + i\frac{\partial^2 \Psi_n(x,\zeta)}{\partial x^2}\right) d\zeta, \ n \ge 0 \quad (23)$$

This yields stationary conditions

$$1+\lambda = 0 \ , \lambda' = 0$$

This in turn gives $\lambda = -1$.

Substituting this value of the Lagrange multiplier into the functional gives the iteration formula.

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) - \int_0^t \left(\frac{\partial \Psi_n(x,\zeta)}{\partial \zeta} + i\frac{\partial^2 \Psi_n(x,\zeta)}{\partial x^2}\right) d\zeta, \ n \ge 0 \quad (24)$$

Selecting $\Psi_0(x,t) = \alpha e^{ikx}$ leads to the successive approximations

$$\Psi_0(x,t) = \alpha e^{ikx}$$

$$\Psi_1(x,t) = (1 + ik^2 t)\alpha e^{ikx}$$

$$\Psi_2(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2\right)\alpha e^{ikx}$$

$$\Psi_3(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2 + + \frac{1}{3!}(ik^2 t)^3\right)\alpha e^{ikx}$$

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$$\Psi_n(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2 + + \frac{1}{3!}(ik^2 t)^3 + \cdots\right)\alpha e^{ikx}$$

This gives the exact solution by

$$\Psi_0(x,t) = \alpha e^{ik(x+kt)} \quad (25)$$

Example 2.
Use the Homotopy perturbation method to solve the linear Schrodinger equation:

$$\frac{\partial \Psi(x,t)}{\partial t} - i\frac{\partial^2 \Psi(x,t)}{\partial x^2} = 0, \quad x \in \mathbf{R}, t > 0 \qquad (26)$$
$$\Psi(x,0) = \sinh x$$

Using HPM, we construct a homotopy in the following form

$$H(\psi, p) = (1-p)\left[\frac{\partial \psi}{\partial t} - \frac{\partial \Psi_0}{\partial t}\right] + p\left[\frac{\partial \psi}{\partial t} - i\frac{\partial^2 \psi}{\partial x^2}\right] = 0 \qquad (27)$$

We select $\Psi_0(x,t) = \sinh x$ as in initial approximation that satisfies the two conditions. Substituting equation (11) into equation (26) end equating the terms with identical powers of p, we drive

$$p^0: \begin{cases} \dfrac{\partial \psi_0}{\partial t} - \dfrac{\partial \Psi_0}{\partial t} = 0, \\ \psi_0(x,0) = \sinh x, \end{cases} \qquad (28)$$

$$p^1: \begin{cases} \dfrac{\partial \psi_1}{\partial t} + \dfrac{\partial \Psi_0}{\partial t} - i\dfrac{\partial^2 \psi_0}{\partial x^2} = 0, \\ \psi_1(x,0) = 0, \end{cases} \qquad (29)$$

$$p^2: \begin{cases} \dfrac{\partial \psi_2}{\partial t} - i\dfrac{\partial \psi_1}{\partial x^2} = 0, \\ \psi_2(x,0) = 0, \end{cases} \qquad (30)$$

$$p^3: \begin{cases} \dfrac{\partial \psi_3}{\partial t} - i\dfrac{\partial \psi_2}{\partial x^2} = 0, \\ \psi_3(x,0) = 0, \end{cases} \qquad (31)$$

$$\vdots$$

Consider $\psi_0 = \Psi_0(x,t) = \sinh x$ Form equations (28), (29) , (30) and (31), we have

$$\psi_1(x,t) = it \sinh x$$

$$\psi_2 = \frac{(it)^2}{2!}\sinh x$$

$$\psi_3 = \frac{(it)^3}{3!}\cosh x$$

Therefore, the solution of equation(26) when $p \to 1$ we will be as follows:

$$\Psi(x,t) = \left(1 + \frac{(it)}{1!} + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right)\cosh x = e^{it}\cosh x \qquad (32)$$

Variational Iteration Method:
By using variational iteration method for equation (16), we obtained

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \int_0^t \lambda(\zeta)\left(\frac{\partial \Psi_n(x,\zeta)}{\partial \zeta} - i\frac{\partial^2 \Psi_n(x,\zeta)}{\partial x^2}\right)d\zeta, \quad n \geq 0 \qquad (33)$$

This yields stationary conditions

$$1+\lambda = 0, \quad \lambda' = 0$$

This in turn gives $\lambda = -1$.

Substituting this value of the Lagrange multiplier into the functional gives the iteration formula.

$$\Psi_{n+1}(x,t)=\Psi_n(x,t)-\int_0^t\left(\frac{\partial\Psi_n(x,\zeta)}{\partial\zeta}-i\frac{\partial^2\Psi_n(x,\zeta)}{\partial x^2}\right)d\zeta,\ \mathrm{n}\geq 0 \tag{34}$$

Selecting $\Psi_0(x,t)=\sinh x$ leads to the successive approximations

$$\Psi_0(x,t)=\sinh x$$

$$\Psi_1(x,t)=\sinh x+it\sinh x$$

$$\Psi_1(x,t)=\sinh x+it\sinh x+\frac{(it)^2}{2!}\sinh x$$

$$\Psi_3(x,t)=\sinh x+it\sinh x+\frac{(it)^2}{2!}\sinh x+\frac{(it)^3}{3!}\sinh x$$

$$\Psi_3(x,t)=\left(1+\frac{it}{1!}+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}\right)\sinh x$$

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$$\Psi_n(x,t)=\left(1+\frac{it}{1!}+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\cdots\right)\sinh x$$

This gives the exact solution by

$$\Psi_n(x,t)=e^{it}\sinh x \tag{35}$$

Example 3

Use the Homotopy perturbation method to solve the linear Schrodinger equation:

$$i\frac{\partial\Psi(x,t)}{\partial t}+\frac{\partial^2\Psi(x,t)}{\partial x^2}+\gamma|\Psi|^2\Psi=0, \tag{36}$$

$$\Psi(x,0)=e^{ikx}$$

Using HPM, we construct a homotopy in the following form

$$H(\psi,p)=(1-p)\left[\frac{\partial\psi}{\partial t}-\frac{\partial\Psi_0}{\partial t}\right]+p\left[\frac{\partial\psi}{\partial t}+i\left(\frac{\partial^2\psi}{\partial x^2}+\gamma\psi^2\overline{\psi}\right)\right]=0 \tag{37}$$

Where $\Psi^2\overline{\Psi}=|\Psi|^2\Psi$ and $\overline{\Psi}$ is conjugate of Ψ.

We select $\Psi_0(x,\mathrm{t})=e^{ikx}$ as in initial approximation that satisfies the two conditions. Substituting equation (11) into equation (36) end equating the terms with identical powers of p, we drive

$$p^0:\ \begin{cases}\dfrac{\partial\psi_0}{\partial t}-\dfrac{\partial\Psi_0}{\partial t}=0,\\ \psi_0(x,0)=e^{ikx},\end{cases} \tag{38}$$

$$p^1:\ \begin{cases}\dfrac{\partial\psi_1}{\partial t}+\dfrac{\partial\Psi_0}{\partial t}+i\left(\dfrac{\partial^2\psi}{\partial x^2}+\gamma\psi^2\overline{\psi}\right)=0,\\ \psi_1(x,0)=0,\end{cases} \tag{39}$$

$$p^2: \begin{cases} \dfrac{\partial \psi_2}{\partial t} - i\left(\dfrac{\partial^2 \psi_1}{\partial x^2} + \gamma(\psi_0^2 \overline{\psi}_1 + 2\psi_1\psi_0\overline{\psi}_0)\right) = 0, \\ \psi_2(x,0) = 0, \end{cases} \tag{40}$$

$$p^3: \begin{cases} \dfrac{\partial \psi_2}{\partial t} + i\left(\dfrac{\partial^2 \psi_1}{\partial x^2} + \gamma(\psi_0^2 \overline{\psi}_2 + \psi_1^2\overline{\psi}_0 + 2\psi_1\psi_0\overline{\psi}_1 + 2\psi_0\psi_2\overline{\psi}_0)\right) = 0, \\ \psi_2(x,0) = 0, \end{cases} \tag{41}$$

$\vdots$

Consider $\psi_0 = \Psi_0(x, t) = e^{ikx}$. Form equations (38), (39) , (40) and (41), we have

$$\psi_1 = (k^2 it)e^{ikx}$$

$$\psi_2 = \frac{(k^2 it)^2}{2!}e^{ikx}$$

$$\psi_3 = \frac{(k^2 it)^3}{3!}e^{ikx}$$

$\vdots$

Therefore , the solution of equation(36) when $p \to 1$ we will be as follows:

$$\Psi(x, t) = \left(e^{ikx} + \frac{(k^2 it)}{1!}e^{ikx} + \frac{(k^2 it)^2}{2!}e^{ikx} + \frac{(k^2 it)^3 e^{ikx}}{3!} + \cdots\right) = e^{ik(x+kt)} \tag{42}$$

Variational Iteration Method:

By using variational iteration method for equation (26), we obtained

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + \int_0^t \lambda(\zeta)\left(i\frac{\partial \Psi_n(x,\zeta)}{\partial \zeta} + \frac{\partial^2 \Psi_n(x,\zeta)}{\partial x^2} + \gamma\ \Psi_n\tilde{\Psi}_n\right)d\zeta,\ \ n \ge 0 \tag{34}$$

Where $|\Psi|^2 = \Psi\tilde{\Psi}$ and $\tilde{\Psi}$ is conjugate of Ψ.

This yields stationary conditions

$$1+i\lambda = 0\ ,\lambda' = 0$$

This in turn gives $\lambda = -i$

Substituting this value of the Lagrange multiplier into the functional gives the iteration formula.

$$\Psi_{n+1}(x,t) = \Psi_n(x,t) + i\int_0^t \left(i\frac{\partial \Psi_n(x,\zeta)}{\partial \zeta} + \frac{\partial^2 \Psi_n(x,\zeta)}{\partial x^2} + \gamma\ \Psi_n\tilde{\Psi}_n\right)d\zeta,\ \ n \ge 0 \tag{35}$$

Selecting $\Psi_0(x,t) = \alpha e^{ikx}$ leads to the successive approximations

$$\Psi_0(x,t) = e^{ikx}$$

$$\Psi_1(x,t) = (1 + ik^2 t)e^{ikx}$$

$$\Psi_2(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2\right)e^{ikx}$$

$$\Psi_3(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2 + + \frac{1}{3!}(ik^2 t)^3\right)e^{ikx}$$

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$$\Psi_n(x,t) = \left(1 + ik^2 t + \frac{1}{2!}(ik^2 t)^2 + + \frac{1}{3!}(ik^2 t)^3 + \cdots\right)e^{ikx}$$

This gives the exact solution by

$$\Psi_0(x,t) = e^{ik(x+kt)} \tag{36}$$

V. CONCLUSION

In this paper, we compared homotopy perturbation and variational iteration method as applied to solve the linear and nonlinear Schrodinger equation. It shown that thesemethod are very efficient and powerful to get the exact solution.Variational iteration method gives several successive approximations through using the iteration of the correction functional, and requires the evaluation of the Lagrangian multiplier.These methods are gives more realistic series solutions that converge very rapidly in physical problems. The Schrodinger equation under the initial conditions, give similar results when we use the homotopy perturbation method and variational iteration method.

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