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An Algorithm to Construct Symmetric Latin Squares of Order q^n for $q \geq 2$ and $n \geq 1$

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ABSTRACT: *Latin squares of order n exist for each n ≥ 1. There are severalways of constructing Latin squares. Also for n≥ 2, if the number of reduced Latin squares isknown, then the number of general Latin squares canbecalculated. This paperproposed a generalmethod to constructsymmetric Latin squares of orderq^n by using blocks of order q which have the basic property of a recursivealgorithmwith the use of cyclicshiftingmethod. Further, the resultingsymmetric Latin squares have the property of reduced Latin squares. The proposedalgorithmwastestedmannualy for q=2,3,4 and 5.For higherorder Latin squares wastestedusing Java progaramm.Thisalgorithmcouldbegeneralized for any q ≥ 2 and n ≥1.*

Keywords: *Combinatorics, Cyclic Shifting, Graph theory, Latin Square, Symmetry*

I. INTRODUCTION

The name “Latin Square” was inspired by mathematical papers by Leonhard Euler (1707-1783), who used Latin characters as symbols or entries, but any set of symbols can be used. He seems to be the first to define them using mathematical terminology, and the first to investigate their properties mathematically. Later, he published them in the paper beginning with the famous 36 officers’ problem (presented to the Academy of Sciences in St. Petersburg in 1779, published in 1782), and then he immediately launched a more complicated concepts. There are several applications based on Latin square in Combinatorics [1], Graph theory [2], Design theory [3], and Statistics [4], and Computer science [5]. The term Latin squares is used in general to describe groups in algebra. They are characterized as multiplication tables or Cayley Tables of Quasi groups [6]. The table of values of a Binary operation that forms a Latin Square obeys the property of the Latin Square. Latin Squares can be used in row-column designs for two blocking factors [7]. Many row column designs are constructed by combining Latin Squares. Also, the popular Sudoku puzzles are a special case of Latin squares; any solution to a Sudoku puzzle is a Latin square.

Definition 1: (Latin Square)
A Latin Square of order n is an n x n matrix containing n different symbols such that each symbol occurs in each row and each column exactly once .[1]
The following is an example of a Latin square of order 4.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$$

Definition 2: (Reduced Latin Square)
A reduced Latin square of order n is a Latin square that has its first row and first column in the natural order. [1]The reduced Latin square of order 4 is given by

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

Definition 3: (Symmetric Latin Square)
Let L be a Latin square of order n. If $L=L^T$, where L^T is the transpose of L, then L is said to be a Symmetric Latin square of order n. [1]
Let A and B be two Latin squares of order 4,where

$$A=\begin{matrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&1\end{matrix}\qquad B=\begin{matrix}1&2&3&4\\4&1&2&3\\3&4&1&2\\2&3&4&1\end{matrix}$$

Then,

$$A^T=\begin{matrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&1\end{matrix}\qquad B^T=\begin{matrix}1&4&3&2\\2&1&4&3\\3&2&1&4\\4&3&2&1\end{matrix}$$

Note that $A=A^T$ whereas $B\neq B^T$. Thus, A is a symmetric Latin square and B is not.

1.1 Mathematical Properties of Latin Squares
1.1.1 Orthogonal Array representation
The set of n^2 triples called orthogonal array representation of a square if each entry of an n x n Latin square can be written as a triple (r, c, s), where r is the row, c is the column, s is the symbol.
Consider the Latin square of order 3

$$\begin{matrix}1&2&3\\2&3&1\\3&1&2\end{matrix}$$

Orthogonal array representation is {(1,1,1), (1,2,2), (1,3,3), (2,1,2), (2,2,3), (2,3,1), (3,1,3), (3,2,1), (3,3,2)}
The definition of Latin square also can be written in term of orthogonal arrays [8]. A squareis a Latin square if the set of all triples (r, c, s) where $1 \leq r, c, s \leq n$, such that all ordered pairs (r, c), (r, s), (c, s) are distinct.
For any Latin square, there are n^2 triples. The orthogonal array representation shows that rows, columns, and symbols play rather similar roles, as will be made below.

1.1.2 Equivalence classes of Latin square
Many operations on a Latin square produce another Latin square. Permutation is one such operation. If we permute columns, permute rows, relabeling symbols of a Latin square, a new Latin square isotopic to the first is obtained.Isotopism is an equivalence relation. The equivalence classes of this relation are called Isotopy classes. [9] Another type of operation is easy to explain by using orthogonal representation. We can replace (r, c, s) by (c, s, r)likewise altogether there are six possibilities. Those are called conjugate (also parastrophes) of the original square.Also we can combine above two equivalence operation to form another equivalence relation. If one of Latin square is isotopic to a conjugate of the other, then those two Latin squares are said to be paratopic, also main class isotopic.

1.1.3 Orthogonality
When superimposing two Latin squares of order n, say L_1 and L_2, we get an n x n array $S_{(L_1,L_2)}$of ordered pairs, where the (i, j)- th entry is defined by $S_{(L_1,L_2)}(i,j) = (L_1(i,j), L_2(i,j))$ for $0 \leq i < n$.If r =$P_{(L_1,L_2)}$is the number of distinct ordered pairsobtained after superimposing L_1 and L_2 then, L_1 and L_2 are said to be r-orthogonal.Consider the example of order 4given by

$$L_1=\begin{matrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{matrix}\qquad L_2=\begin{matrix}1&2&3&4\\3&4&1&2\\4&1&2&3\\2&3&4&1\end{matrix}$$

$$S_{(L_1,L_2))}=\begin{matrix}(1,1)&(2,2)&(3,3)&(4,4)\\(2,3)&(3,4)&(4,1)&(1,2)\\(3,4)&(4,1)&(1,2)&(2,3)\\(4,2)&(1,3)&(2,4)&(3,1)\end{matrix}$$

Here r=12. Thus L_1 and L_2 are 12-orthogonal.
Two Latin squares of order n are orthogonal if these squares are superimposed each of the n^2 ordered pairs appears exactly once, that is they are n^2-orthogonal.

1.1.4Automorphisms
An automorphism is an isomorphism [10] from a Latin square to itself. The automorphism of a Latin square form a group. [11]

1.1.5 Partial Latin squares

A partial Latin square of order n is an n x n array in which each cell is filled by either a symbol from the set {1,2,3,…,n} or a blank, so that no symbol is repeated twice in any row or column. It can be converted to a Latin square if it is possible to place symbols in the blank cells.[12] Also, there are partial Latin squares that cannot be converted to a Latin square.

Consider the following A and B.

$$A=\begin{matrix} - & - & - & 4 \\ 2 & - & - & - \\ 3 & 4 & - & - \\ 4 & 1 & 2 & - \end{matrix} \quad B=\begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 2 \end{matrix}$$

(A) can be converted to a Latin Square as
$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix}$$

But (B) cannot be converted to a Latin square.

1.1.6 Transversals

A transversal to a Latin square of order n is a set of n cells which has the property that one cell lies in each column, one in each row, and one contains each entry. A Latin square has an orthogonal mate if and only if it contains n disjoint transversals that partition the n^2 cells of the squares [13].

1.2CyclicShiftingMethod

Write the symbols in the top row in any order. In the second row, shift all the symbols to the left one place, moving the first symbol to the last. Continue like this, shifting each row one place to left of the previous row [14].

The following is a matrix obtained from cyclic shifting method.

$$\begin{matrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{matrix}$$

This method can also be used to shift all the symbols to the right one place.

II. LITERATUREREVIEW

In the past decades, many researchers have developed theories, algorithms to construct some properties of Latin squares, solutions procedures for varies types of Latin squares.Jayathilakeetalintroduced an algorithm to construct super-symmetric Latin squares of order 2^n. Itexplains a set of significant steps that works as a recursive algorithm and the construction of those steps together in to one composite algorithm which ultimately provide the capacity of general super-symmetric Latin squares of order 2^n. The method she used to propose an algorithm is first she constructed the first row of the Latin square by placing the integers in increasing order. Then she separated pairs and interchanges the elements in the pairs in second row. After that she separated rows into $4(2^2)$ blocks and interchange these blocks and continued this process separating rows into $8(2^3)$, $16(2^4)$ … until she had a Latin square corresponding to the order.

Kartika etal proposed a new algorithm to construct super-symmetric Latin squares of order 2n. In this method he obtained 2 Latin squares of order n by cyclic shifting opposite directions called as blocks and placed them in first row. Then interchange those blocks in order to obtain second row.

Allan B Cruse etal is used to find on embedding incomplete symmetric Latin squares. Cruse described necessary and sufficient conditions are obtained for the extendibility of t x t symmetric Latin rectangle to an n x n Symmetric Latin square. Conditions imply that any incomplete n x n symmetric Latin square can be embedded in a complete symmetric Latin square of order 2n. Also, he discoveredthat any incomplete n x n symmetric diagonal Latin square can be embedded in a complete symmetric diagonal Latin Square of order 2n+1.

Bailey et al discussed about construction and randomization of quasi-complete Latin square. A general method for constructing quasi-complete Latin squares based on group is given. This method leads to a relatively straightforward way of counting the number of in equivalent quasi-complete Latin squares of order 9. Randomization of such designs is discussed, and an explicit construction for valid randomization sets of quasi-complete Latin squares whose side is an odd prime power is presented.

III. METHODOLOGY

The methodology of formulating symmetric Latin square of order q^n, where $q \geq 2$, $n \in \mathbb{Z}^+$ which is explained under this paper affiliates a set of compelling steps that works as a recursive algorithm. This section explains the construction of those steps together in to one composite algorithm which ultimately provide the capability of establishing symmetric Latin squares of order q^n.

3.1 Steps of the Proposed Algorithm

1. When q=2;

- **When n=1,**

Let B_1^1=1 and B_1^2= 2. Use cyclic shifting method to obtain 2 x 2 symmetric Latin square.

$$B_2^1 = \begin{bmatrix} B_1^1 & B_1^2 \\ B_1^2 & B_1^1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 \\ 2 & 1 \end{bmatrix}$$

Tab: 1

- **When n=2,**

That is the order of Latin square is 2^2. Replace the numbers 1, 2 with numbers 3, 4 in B_2^2.Use cyclic shifting method to obtain the next symmetric Latin square.

$$B_3^1 = \begin{bmatrix} B_2^1 & B_2^2 \\ B_2^2 & B_2^1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \end{bmatrix}$$

Tab: 2

2. When q=3

- **When n=1,**

Let B_1^1=1, B_1^2= 2 and B_1^3=3. Use cyclic shifting method to obtain 3x3 symmetric Latin square.

$$B_2^1 = \begin{bmatrix} B_1^1 & B_1^2 & B_1^3 \\ B_1^2 & B_1^3 & B_1^1 \\ B_1^3 & B_1^1 & B_1^2 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Tab: 3

- **When n=2,**

That is the order of Latin square is 3^2. Replace the numbers 1, 2, 3 with numbers 4, 5, 6 in B_2^2. Replace the numbers 1, 2, 3 with numbers 7, 8, 9 in B_2^3 .Use cyclic shifting method to obtain the next symmetric Latin square.

$$B_3^1 = \begin{bmatrix} B_2^1 & B_2^2 & B_2^3 \\ B_2^2 & B_2^3 & B_2^1 \\ B_2^3 & B_2^1 & B_2^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{bmatrix} & \begin{bmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \\ 9 & 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{bmatrix} & \begin{bmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \\ 9 & 7 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \\ 9 & 7 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{bmatrix} \end{bmatrix}$$

Tab: 4

IV. RESULTS AND DISCUSSION

It can be seen that if the symmetric Latin square of order 2 is known, then 2^n symmetric Latin square can be constructed using proposed algorithm.

When n=N, The corresponding Latin square will be;

$$B_{N+1}^1 = \begin{bmatrix} B_N^1 & B_N^2 \\ B_N^2 & B_N^1 \end{bmatrix}$$

WhereB_N^1 consists of 1, 2… $2^{(N-1)}$ andB_N^2 consists of $2^{(N-1)}$ +1, $2^{(N-1)}$ +2,…, 2^N

A considerable amount of symmetric Latin squares of order q^nwere constructed using the proposed algorithm.

Fig. 01 displays the symmetric Latin square of order 2 which were generated by the computerized version (Java) of the proposed algorithm.

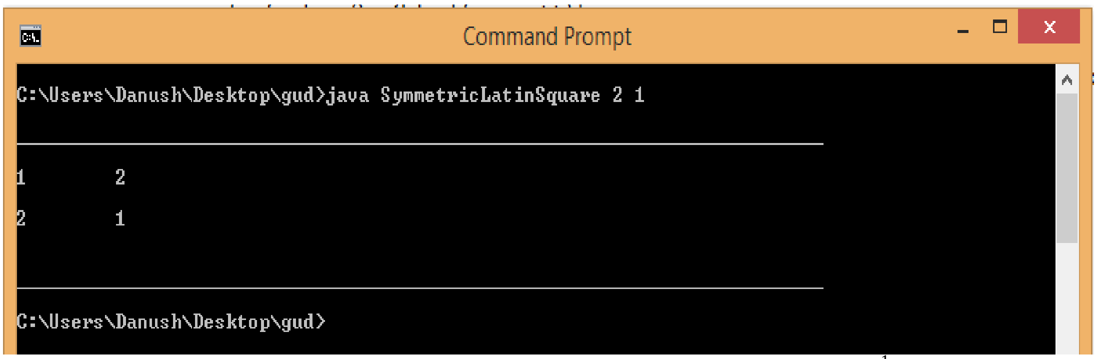


Fig.01-computer generated symmetric Latin square of order 2(2^1)

Symmetric Latin square can be plotted using the MATLAB plot function which generates a specific pattern. Fig. 02 shows the plotted graphs of symmetric Latin square of orders 2 (2^1).

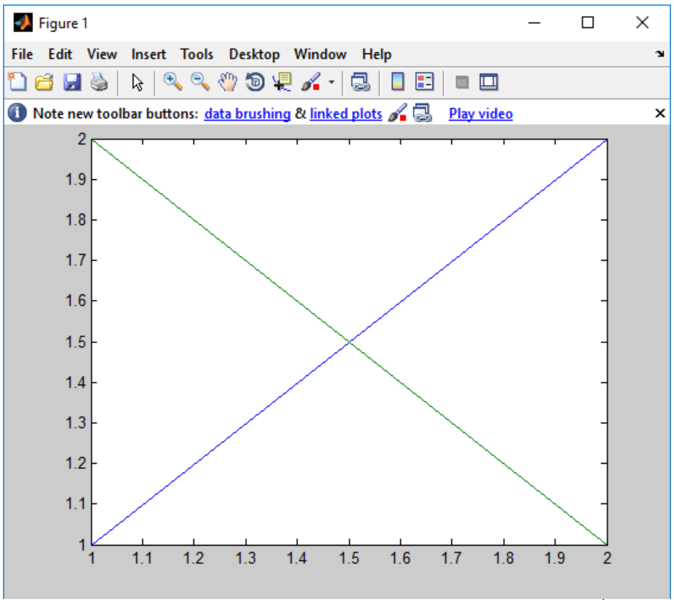


Fig.02-plotted symmetric Latin square of order 2(2^1)

Fig. 03 displays the symmetric Latin square of order 4 which were generated by the computerized version(Java) of the proposed algorithm.

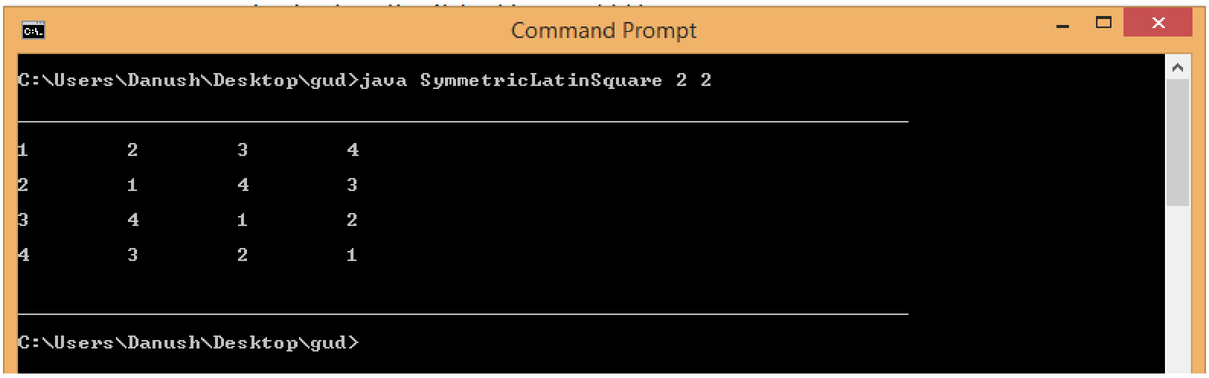


Fig.03-computer generated symmetric Latin square of order 4(2^2)

Fig. 04 shows the plotted graphs of symmetric Latin square of orders 4 (2^2).

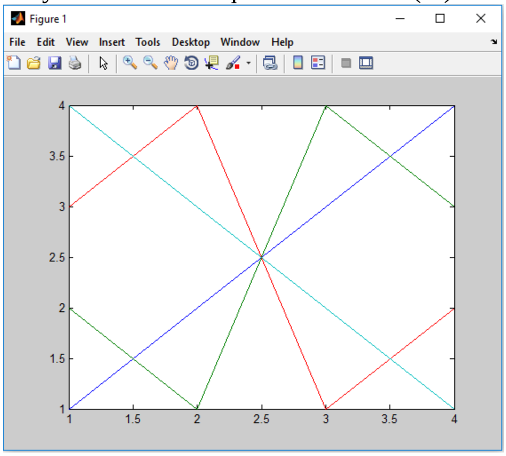


Fig.04-plotted symmetric Latin square of order 4(2^2)

It can be seen that if the symmetric Latin square of order 3 is known, then 3^n symmetric Latin square can be constructed using proposed algorithm.
When n=N, the corresponding Latin square will be;

$$B_{N+1}^{1} = \begin{vmatrix} B_N^1 & B_N^2 & B_N^3 \\ B_N^2 & B_N^3 & B_N^1 \\ B_N^3 & B_N^1 & B_N^2 \end{vmatrix}$$

Where B_N^1 consists of $1,2,\ldots,3^{(N-1)}$, B_N^2 consists of $3^{(N-1)}+1, 3^{(N-1)}+2,\ldots,2x3^{(N-1)}$ and B_N^3 consists of $2x3^{(N-1)}+1, 2x3^{(N-1)}+2,\ldots,3^N$

Fig. 05displays the symmetric Latin square of order 3 which were generated by the computerized version (Java) of the proposed algorithm.

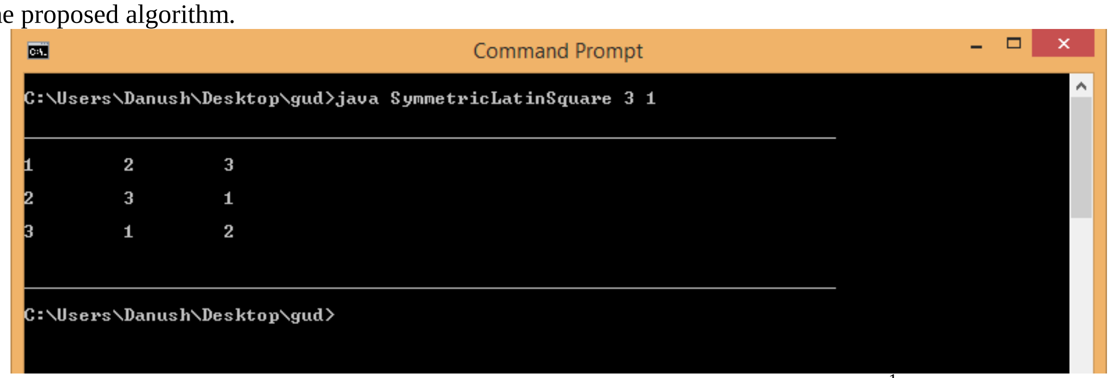


Fig.05-computer generated symmetric Latin square of order 3(3^1)

Fig. 06 shows the plotted graphs of symmetric Latin square of orders 3 (3^1) using MATLAB plot function.

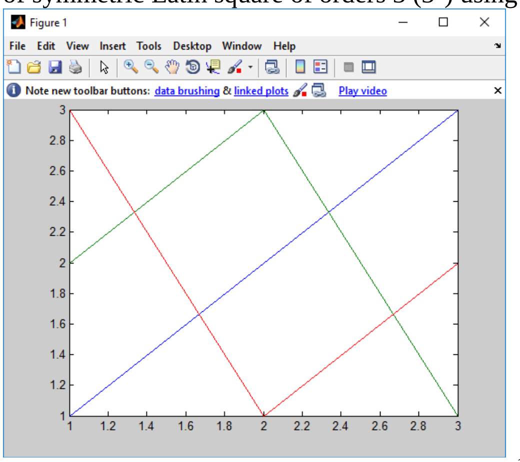


Fig.06-plotted symmetric Latin square of order 3(3^1)

Fig. 07 displays the symmetric Latin square of order 9 which were generated by the computerized version (Java) of the proposed algorithm.

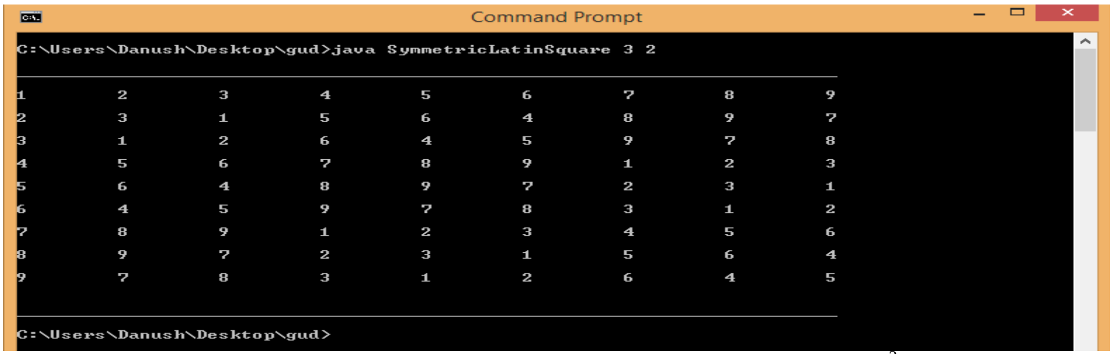


Fig.07-computer generated symmetric Latin square of order 9(3^2)

Fig. 08 shows the plotted graphs of symmetric Latin square of orders 9 (3^2) using MATLAB plot function.

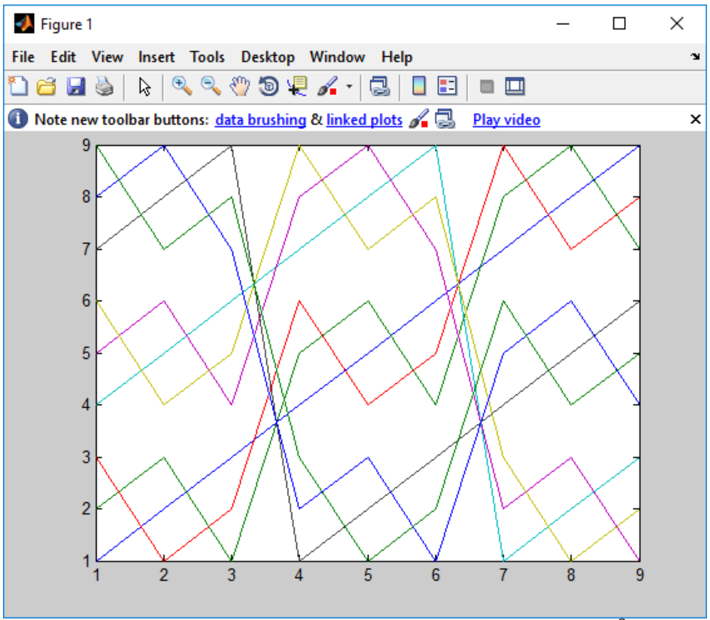


Fig.08-plotted symmetric Latin square of order 9(3^2)

Considering q=4, the resulting symmetric Latin Square of order 4 and 16 were generated utilizing the proposed algorithm. Since the obtained Latin squares have its first row and first column in the natural order, those are reduced Latin squares.

Fig.09 displays the symmetric Latin square of order 4 which were generated by the computerized version (Java) of the proposed algorithm.

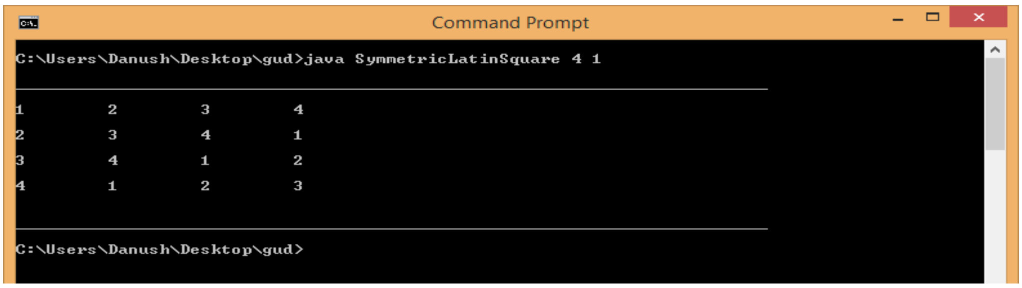


Fig.09-computer generated symmetric Latin square of order 4(4^1)

Fig.10 shows the plotted graph of symmetric Latin square of order $4(4^1)$ using MATLAB plot function.

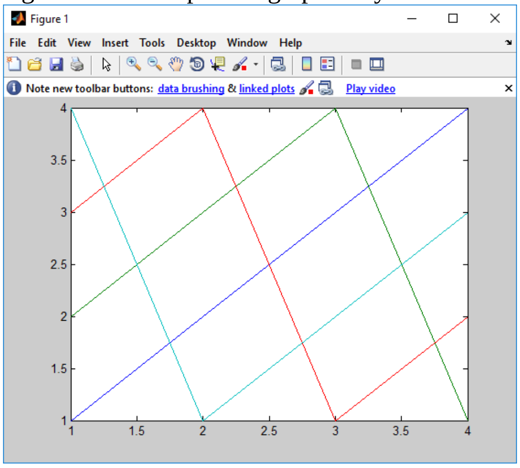
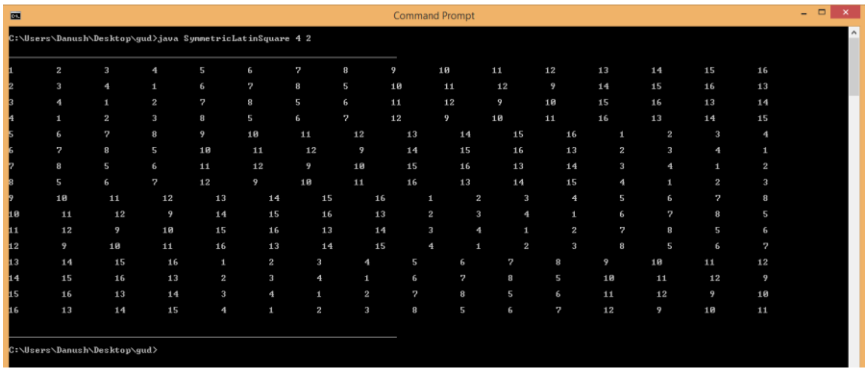


Fig.10-plotted symmetric Latin square of order $4(4^1)$

Fig.11 displays the symmetric Latin square of order 16 which were generated by the computerized version (Java) of the proposed algorithm.


Command Prompt

C:\Users\Danush\Desktop\gud>java SymmetricLatinSquare 4 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	1	6	7	8	5	10	11	12	9	14	15	16	13
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	1	2	3	8	5	6	7	12	9	10	11	16	13	14	15
5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4
6	7	8	5	10	11	12	9	14	15	16	13	2	3	4	1
7	8	5	6	11	12	9	10	15	16	13	14	3	4	1	2
8	5	6	7	12	9	10	11	16	13	14	15	4	1	2	3
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	11	12	9	14	15	16	13	2	3	4	1	6	7	8	5
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	9	10	11	16	13	14	15	4	1	2	3	8	5	6	7
13	14	15	16	1	2	3	4	5	6	7	8	9	10	11	12
14	15	16	13	2	3	4	1	6	7	8	5	10	11	12	9
15	16	13	14	3	4	1	2	7	8	5	6	11	12	9	10
16	13	14	15	4	1	2	3	8	5	6	7	12	9	10	11

C:\Users\Danush\Desktop\gud>

Fig.11-computer generated symmetric Latin square of order $16(4^2)$

Fig.12 shows the plotted graph of symmetric Latin square of order $16(4^2)$ using MATLAB plot function.

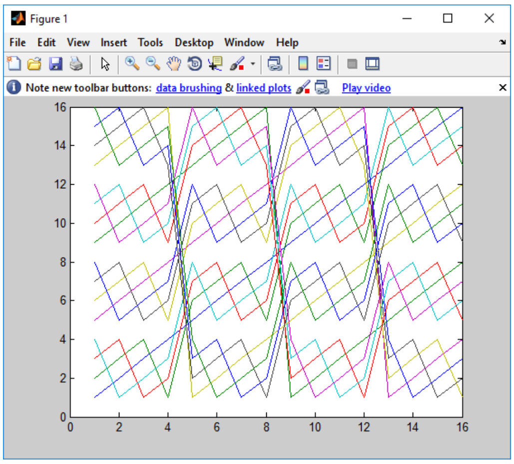


Fig.12-plotted symmetric Latin square of order $16(4^2)$

Considering q=5, the resulting symmetric Latin Square of order 5, 25, 125... can be generated utilizing the proposed algorithm. Since the obtained Latin squares have its first row and first column in the natural order, those are reduced Latin squares.

Fig.13 displays the symmetric Latin square of order 5 which were generated by the computerized version (Java) of the proposed algorithm.

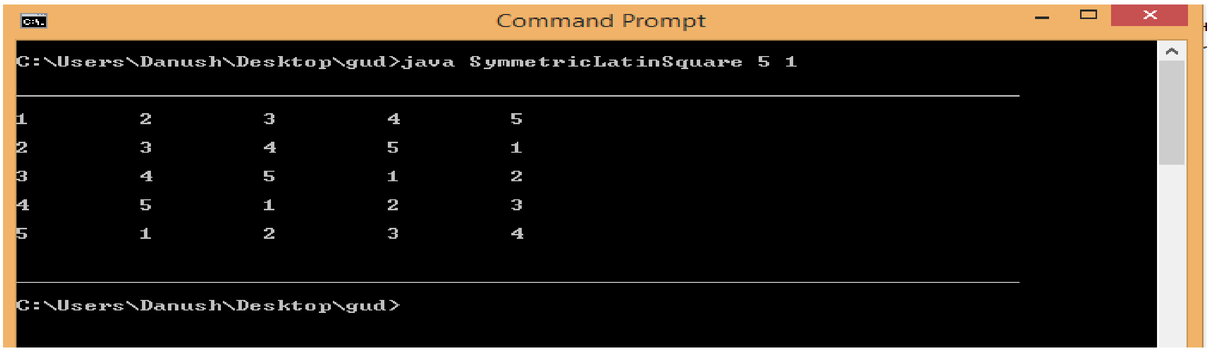


Fig.13-computer generated symmetric Latin square of order $5(5^1)$

Fig.14 shows the plotted graph of symmetric Latin square of order $5(5^1)$ using MATLAB plot function.

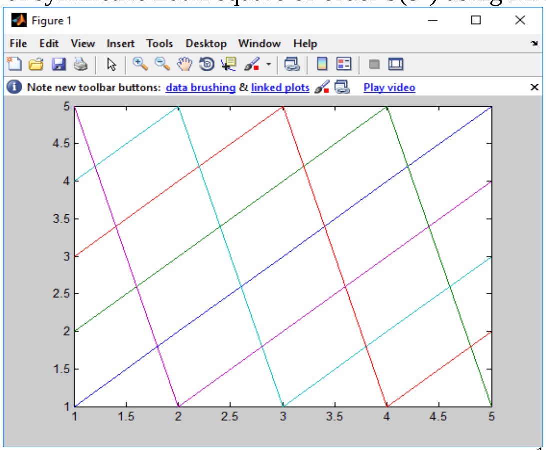


Fig.14-plotted symmetric Latin square of order $5(5^1)$

Therefore, the similar method can be used to obtain the symmetric Latin squares of order q^n $for\ q \geq 2\ and\ n \geq 1$

V. Conclusion

A mathematical method for generating symmetric Latin squares was introduced. Test cases provided that the proposed algorithm can be generalized. When the order was small, construction can be established manually, and the algorithm has been automated using simple steps. Further, this method can be extended to construct the super- symmetric Latin square of order $2q^n$ where q, n ≥ 1.

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