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Comparative Analyses of Iteratively Generated Fractals: The Cases of Two "Chaos Game" Options

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ABSTRACT: *Several research efforts has been made toward establishing that the decimal digits of the mathematical constant Pi (π) follows each other in a random sequence. The present study have same objective from visual perspective involving qualitative and quantitative analyses of iteratively generated fractals using two "Chaos Game" implementations. Six fractals images were generated in pair from their corresponding affine functions using Pseudo-randomly generated numbers and Pi digits respectively as selection criteria. The variations in visual and average improved fractal disk results between the paired fractals were found to be insignificant thereby establishing those Pi digits follow each other sufficiently in a random sequence.*

Keywords: *Pi, Random, Iteratively Generated Fractals, Affine function and Chaos Game*

I. INTRODUCTION

A mathematical constant Pi is an irrational number with continuous decimal digits part but often approximated as 3.142. It is generally represented by the symbol π and explored on daily basis for estimating the circumference of circles. It has been well described in literature as the ratio of circle's circumference to its diameter (Xiong, 2010; Bartholomew, 2014; Didur, 2014 and KU, 2015).The importance of Pi in engineering is enormous and cannot be over emphasized. In the design of machines and other engineering components or products such as camshaft, crankshaft, piston, engine block, radiators, tires, pipeline, conduit and cabling systems, boilers, bolt and nuts, round gaskets and oil tankers just to mention a few, the use of Pi as part of estimating parameters is quintessential. Sequel to the numerous applications and long time significance of Pi, several laudable research efforts (Gibbs, 2003; Gibbons, 2005; Bailey *et al*, 2013; Pampena, 2013 and Borwein *et al*, 2015) have been made on this mathematical constant and more especially on its forever occurring decimal digit part.

A reasonable number of works dwell on the randomness of Pi decimal digits. For instance, Bailey (1988) performed statistical analysis on the expansion of Pi to 29,360,000 decimal places and found that frequencies of n-long strings of digits for n up to 6 are completely unremarkable. It was concluded in the paper that the decimal expansion of π appears to be entirely arbitrary. Sourabh *et al*, (2009) studied the randomness of the subsequences of Pi decimal digits. The authors found that randomness exist for the first 960 million digits and suspected to be generally arbitrary for other subsequences of the number decimal digits. Khodabin (2011) investigated the statistical properties of Pi number decimal digits. The first 40960 digits of π-3 were considered and results showed that all digits distributed uniformly in 40960 decimal digits. The author was able to establish that transmission between all digits in one step is wholly random. It was concluded that a statistical model for reliable prediction of next digit in decimal digits of Pi is yet to exist.

Despite robust efforts that have been made on the study of randomness in respect of decimal digits part of Pi, the literature is still sparse on investigating the arbitrariness of these digits of the mathematical constant using fractal concept. This work focuses on qualitative (the visual evaluation) as well as quantitative (the statistical distribution of the estimated improved disk dimensions) analyses of iteratively generated fractals using Chaos Games implemented respectively by randomly generated numbers and Pi decimal digits.

The definition of fractal well reported in the literature and can be described as a rough or fragmented geometric shape that can be subdivided into parts, each of which is (at least approximately) a reduced size or self-similar copy of the whole(Rama and Mishra, 2010 and Singh *et al*, 2012). The term was well reported in the literature to be coined by Benoit Mandelbrot in 1975 and was derived from the Latin word "fractus" implying "broken" or "part" (Lopes, 2009; Rama and Mishra, 2010 and Singh *et al*, 2012). The basic properties of fractals are self-similarity, scale invariance and general irregularity in shape with more significant details at higher magnifications (Rama and Mishra, 2010).

Fractal has been widely used as a resourceful concept for characterization (and modelling) of shapes or structures in science and engineering based research problems (Magana *et al*, 2004; Camps-Raga and Islam, 2010; Khanbareh, 2011; Kumar, 2013 and Salau *et al*, 2016)). Iteratively generated fractal is gaining prominence for qualitative and quantitative analyses from visual perspective (Singh *et al*, 2012; Chugh and Ashish, 2014 and Kamal, 2015). The use of superior iteration methods for implementing two-step feedback systems was the beginning of new iterative technique in fractal modelling (Singh *et al*, 2012) and has been found to have many potential applications. Emerging fractals for various equations have been explored using one-step, two-step, three-step and four-step iterative techniques (Chugh and Ashish, 2014). The foregoing was a great motivation for the adoption of iteratively generated fractals in the present study (using randomly generated numbers and Pi decimal digits as tools for implementation of the Chaos game). The outcome of this investigation can lead to generation of some affine oriented fractals which in turn can revolutionize the existing engineering applications of Pi.

II. METHODOLOGY

The present study adopted basically the procedure outlined by Salau *et al* (2016) for both fractal generation using "Chaos Game" algorithms and fractal characterization. However in the present study, there are two game options; "Game1" involves generation of Pseudo-random number (μ) with a seed of 9876 returning values that spread continuously between ($0 \le \mu \le 1.0$) while "Game2" was based on reading and checking sequential decimal digits of π in which returning values are discrete integer between (0-9). The affine function with matching determinant (see equations 1 and 2) as the returned value in each of the game options is then selected preferentially to advance the fractal generation by additional one simulation step till sufficient steps are obtained. According to Edward (1996), an affine is defined completely when value are given to (a, b, c d, e and f) in the iterative equation (1) while the frequency of a given affine playing the "Chaos Game" among other competing affine can be estimated by determinant (DET) equation (2). Using least square algorithms, the best slope (D_f) of log-log plots of disk size (X) and the minimum number of disks (Y) needed to cover fractal image completely were determined as equivalent to the improved fractal disk dimension (D_f) from the power law proportionate' equation (3).

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}_n + \begin{Bmatrix} e \\ f \end{Bmatrix} \qquad (1)$$

$$DET = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\| = (ad - bc) \qquad (2)$$

$$Y \alpha X^{D_f} \qquad (3)$$

Simulation Parameters

Six different cases of fractals (A, B, C, D, E and F) with corresponding affine functions given in tables 1 to 6 were simulated from common initial coordinate (1, 2) and through unsteady (first 100 solution points) and steady (next 2000 solution points) that were arbitrarily selected. The steady solutions was thereafter characterized using five iteration levels and ten different disk sizes as recommended by Salau *et al* (2016).

Table 1: Affine function parameters and their selection conditions for Case A

Affine No (Case A)	Affine function parameters						Affine selection conditions	
	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Any-of)
1	0.5000	0.0000	0.0000	0.5000	0.0000	0.0000	$0 \le \mu \le 0.33$	0, 1 & 2
2	0.5000	0.0000	0.0000	0.5000	0.5000	0.0000	$0.33 \le \mu \le 0.67$	3, 4 & 5
3	0.5000	0.0000	0.0000	0.5000	0.2500	0.5000	$0..67 \le \mu \le 1.00$	6, 7 & 8

Table 2: Affine function parameters and their selection conditions for Case B

	Affine function parameters						Affine selection conditions	
Affine No (Case B)	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Any-of)
1	0.3333	0.0000	0.0000	0.3333	0.0000	0.0000	$0 \le \mu \le 0.25$	0 & 1
2	0.1667	-0.2887	0.2887	0.1667	0.3333	0.0000	$0.25 \le \mu \le 0.55$	2 & 3
3	0.1667	0.2887	-0.2887	0.1667	0.5000	0.2887	$0.55 \le \mu \le 0.75$	4 & 5
4	0.3333	0.0000	0.0000	0.3333	0.6667	0.0000	$0.75 \le \mu \le 1.00$	6 & 7

Table 3: Affine function parameters and their selection conditions for Case C

		Affine function parameters					Affine selection conditions	
Affine No (Case C)	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Any-of)
1	0.3333	0.0000	0.0000	0.3333	0.0000	0.0000	$0 \le \mu \le 0.20$	0 & 1
2	0.3333	0.0000	0.0000	0.3333	0.6667	0.0000	$0.20 \le \mu \le 0.40$	2 & 3
3	0.3333	0.0000	0.0000	0.3333	0.0000	0.6667	$0.40 \le \mu \le 0.60$	4 & 5
4	0.3333	0.0000	0.0000	0.3333	0.6667	0.6667	$0.60 \le \mu \le 0.80$	6 & 7
5	0.3333	0.0000	0.0000	0.3333	0.3333	0.3333	$0.80 \le \mu \le 1.00$	8 & 9

Table 4: Affine function parameters and their selection conditions for Case D

		Affine function parameters					Affine selection conditions	
Affine No (Case D)	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Is)
1	0.3333	0.0000	0.0000	0.3333	0.0000	0.0000	$0 \le \mu \le 0.13$	0
2	0.3333	0.0000	0.0000	0.3333	0.3333	0.0000	$0.13 \le \mu \le 0.25$	1
3	0.3333	0.0000	0.0000	0.3333	0.6667	0.0000	$0.25 \le \mu \le 0.38$	2
4	0.3333	0.0000	0.0000	0.3333	0.0000	0.3333	$0.38 \le \mu \le 0.50$	3
5	0.3333	0.0000	0.0000	0.3333	0.6667	0.3333	$0.50 \le \mu \le 0.63$	4
6	0.3333	0.0000	0.0000	0.3333	0.0000	0.6667	$0.63 \le \mu \le 0.75$	5
7	0.3333	0.0000	0.0000	0.3333	0.3333	0.6667	$0.75 \le \mu \le 0.88$	6
8	0.3333	0.0000	0.0000	0.3333	0.6667	0.6667	$0.88 \le \mu \le 1.00$	7

Table 5: Affine function parameters and their selection conditions for Case E

	Affine function parameters						Affine selection conditions	
Affine No (Case E)	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Any-of/ Is)
1	0.3333	-0.3333	0.3333	0.3333	0.3333	0.0000	$0 \le \mu \le 0.29$	0, 1 & 2
2	0.6667	0.0000	0.0000	0.6667	0.3333	0.3333	$0.29 \le \mu \le 0.89$	3, 4, 5, 6, 7 & 8
3	0.3333	0.0000	0.0000	0.3333	0.6667	0.0000	$0.89 \le \mu \le 1.00$	9

Table 6: Affine function parameters and their selection conditions for Case F

	Affine function parameters						Affine selection conditions	
Affine No (Case F)	a	b	c	d	e	f	Pseudo-random (μ)	Pi-digits (Any-of/ Is)
1	0.2500	0.0000	0.0000	0.2500	0.0000	0.7500	$0 \le \mu \le 0.10$	0
2	0.2500	0.0000	0.0000	0.2500	0.2500	0.5000	$0.10 \le \mu \le 0.20$	1
3	0.2500	0.0000	0.0000	0.2500	0.5000	0.7500	$0.20 \le \mu \le 0.30$	2
4	0.2500	0.0000	0.0000	0.2500	0.7500	0.5000	$0.30 \le \mu \le 0.40$	3
5	0.7500	0.0000	0.0000	0.5000	0.0000	0.0000	$0.40 \le \mu \le 1.00$	4, 5, 6, 7, 8 & 9

III. RESULTS AND DISCUSSION

Figure 1 is the scattered plots obtained from the first 2000 steady simulated solutions (coordinate points) for the Sierpinski triangle (Case A) by the two "Chaos Game" options. Therefore the visual results are qualitatively the same and agreed with literature result. Similar visual agreement were made between the scatter plots of steady solutions (Game1 and Game2 options) for the remaining five cases, but were not presented only to conserve space. The agreement between these corresponding results is evidence supporting the presence of randomness in sequential appearance of the digits after the decimal part of constant Pi (π). Furthermore, the quantitative improved disk dimensions obtained from the studied six cases were as presented in figures 2, 3 and 4. All these results are qualitatively and quantitatively interchangeable between the two game options thereby reinforcing further the randomness of occurrence of the decimal digits of the constant Pi.

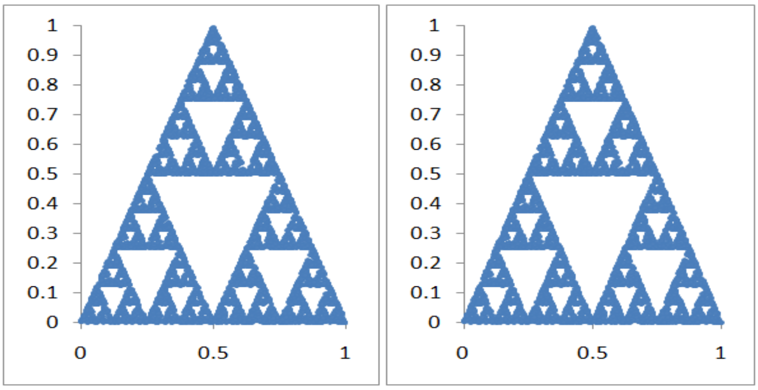

Figure 1: Scattered plots of first 2000 steady simulated solution coordinate points for case A

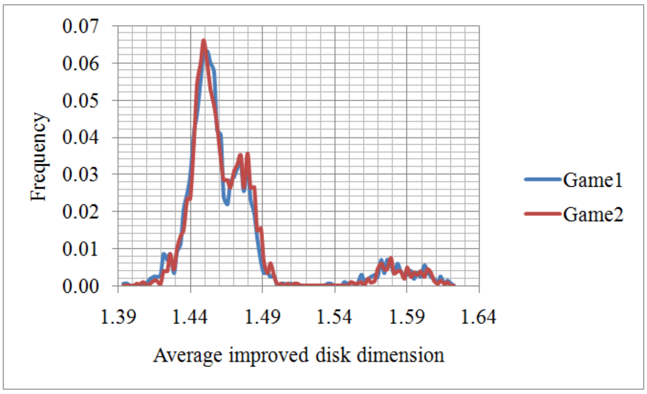

Figure 2: Comparison of distributed average estimated improved disk dimension (D_f) using two game options for the first 2000 steady solution points into 100 equal parts between $1.394 \le D_f \le 1.623$ for case A only.

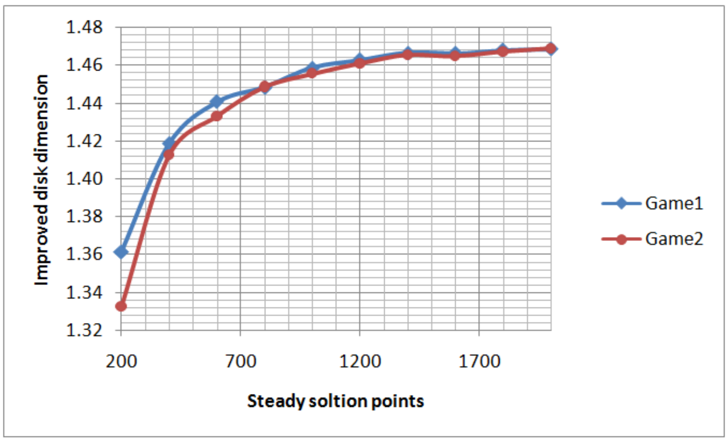

Figure 3: Variation of estimated improved disk dimension with increase in number of steady solution points for case A only

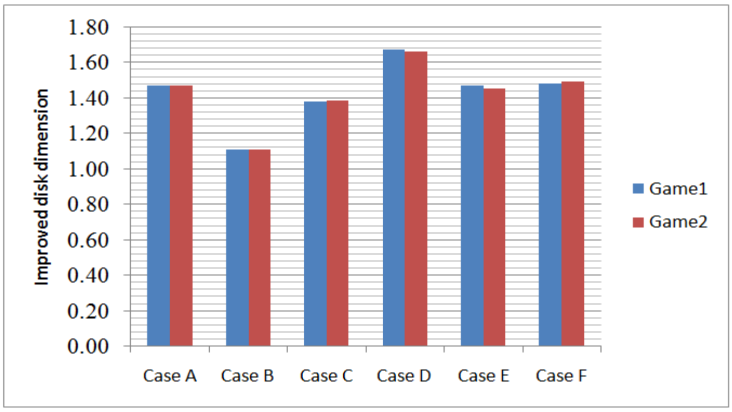

Figure 4: Comparison of the estimated mean value of improved disk dimension using game options (Game1 and Game2) for the first 2000 steady solution points each of cases A to F.

IV. CONCLUSIONS

This study has shown that qualitatively and quantitatively results variations are insignificant for "Chaos Game" played either by pseudo-randomly generated numbers or the consecutive decimal digits of the mathematical constant Pi (π). It is therefore concluded that the string sequence of the decimal digits of Pi (π) is sufficiently random and can be used interchangeably with the conventional pseudo-random numbers to generate some affine oriented fractals for engineering applications.

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