

## Analysis of an Improved SIRS Epidemic Model with Disease Related Death Rate and Emigration Rate

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**Abstract:** In this paper, we consider an SIRS epidemic model with an asymptotically homogeneous transmission function, disease related death rate and emigration rate. We obtain the disease free and endemic equilibrium. We also establish the conditions for the global stability of the equilibriums. An example is also furnished which demonstrates validity of main result.

**Keywords:** SIRS model, Transmission function, Basic reproductive number, disease-free equilibrium, endemic equilibrium, Stability.

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### I. Introduction

A mathematical model is a description of a system using mathematical concept and language. Mathematical models are used not only in the natural sciences and engineering disciplines but also in the social sciences. The first SIR epidemic model was proposed by Kermack and Mckendrick [23] in the year 1927. The SIRS epidemic model has been studied by many authors (see [1-5], Hethcote [16, 17], Capasso and Serio [9], Mena-Lorca [29]) and the different epidemic models have been proposed and studied in the literature ( see Hethcote and Tudar [19], Lie et al. [25, 26], Hethcote et al. [20], Hethcote and Van den Driessche [21], Derrick and Vanden Driessche [12], Berretta and Takeuchi [6, 7], Beretta et al. [8], Ma et al. [27, 28], Ruan and Wang [32], Song and Ma [33], Song et al. [34], D'onofo et al. [13], Xiao and Ruan [35]).

Bilinear and standard incidence rates have been frequently used by many authors [18, 10, 29, 30, 24, 14 and 22]. Disease transmission is a dynamical process driven by the interaction between the susceptible and the infective. The behaviour of the SIRS models are greatly affected by the way in which transmission between infected and the susceptible individuals are modelled. Many models of epidemiology are based on the so called "mass action" assumption for transmission. During the last few decades, such assumptions have faced some questions and consequently a number of realistic transmission functions have become the focus of considerable attention (Capasso and Serio [9], Lie et al. [25, 26], Hethcote et al. [20], Hethcote and Van den Driessche [21], Ruan and Wang [32], Xiao and Ruan [35]). Pathak et al. [31] have considered an SIR epidemic model with an asymptotically homogeneous transmission function.

In this paper we consider an SIRS epidemic model with an asymptotically homogeneous transmission function, disease related death rate and emigration rate. In the next section, we give basic definitions. In the third section, we present the model and derive the disease free equilibrium and the endemic equilibrium. In the fourth section, we prove some theorems for the global stability of the disease free and endemic equilibrium. The fifth section contains an example which demonstrates validity of main result. In the last section, we give conclusion.

### II. Preliminaries

**Definition 2.1** The incidence in an epidemiological model is the rate at which susceptible become infectious. If the unit time is days, then the incidence is the number of new infection per day.

**Definition 2.2** The average number of secondary infections produced by one infected individuals during the mean course of infection (infectious period) in a completely susceptible population is called a basic reproductive number or simply the reproductive number  $\sigma$ .

**Definition 2.3** SIRS means the recovered individuals have only temporary immunity after they recovered from infection.

### III. The mathematical model

The proposed model is the nonlinear ordinary differential equations:

$$\begin{aligned}\frac{dX}{dt} &= (A - B) - \frac{\beta XY}{1 + aX + bY} - dX + \delta Z \\ \frac{dY}{dt} &= \frac{\beta XY}{1 + aX + bY} - (\gamma + \alpha + d)Y \\ \frac{dZ}{dt} &= \gamma Y - (\delta + d)Z \\ \frac{dN}{dt} &= (A - B) - dN - \alpha Y\end{aligned}\quad (3.1)$$

Where  $N(t)$  is the total varying population size as a function of time  $t$ , and  $X(t), Y(t), Z(t)$  denote the number of individuals who are susceptible, infectious and recovered at time  $t$ , respectively and  $X(t) + Y(t) + Z(t) = N(t)$ ,  $A$  is the constant immigration rate of the population,  $B$  is the emigration rate of the population,  $d$  is the natural death rate of the population,  $\beta$  is the transmission coefficient,  $\alpha$  is the disease-related death rate constant,  $\gamma$  is the natural recovery rate of the infective individuals,  $\delta$  is the loss of immunity rate constant,  $a$  and  $b$  are the parameters which measure the effects of sociological, psychological or other mechanisms. We assume that  $d, \alpha$  and  $\delta$  are nonnegative and that  $A, B, \beta, \gamma$  and  $\delta + d$  are positive.

Where  $N = X + Y + Z$ . In the absence of disease i.e.  $\alpha = 0$  the population size approaches the constant size  $\frac{A - B}{d}$ . For the asymptotically transmission function the contact number or basic reproduction number is

$$\sigma = \frac{\beta(A - B) - (A - B)a(\gamma + \alpha + d)}{d(\gamma + \alpha + d)}.\quad (3.2)$$

For the system (3.1) the first octant in  $XYZ$  space is positively invariant. Because  $\frac{dN}{dt} < 0$  for  $N > \frac{A - B}{d}$ , all paths in the first octant approach, enter or stay inside the subset

$$T = \left\{ (X, Y, Z) : X + Y + Z \leq \frac{A - B}{d} \right\}.$$

The continuity of the right side of (3.1) and its derivatives implies

that unique solutions exists on a maximal time interval. Since solutions approach, enter or stay in  $T$ , they are eventually bounded and hence exist for all positive time [11]. We first consider the existence of equilibrium of system (3.1).

For any values of parameter, model (3.1) always has a disease-free equilibrium  $P_o = \left( \frac{A - B}{d}, 0, 0 \right)$ . To find

the positive equilibria, set

$$\begin{aligned}(A - B) - \frac{\beta XY}{1 + aX + bY} - dX + \delta Z &= 0 \\ \frac{\beta XY}{1 + aX + bY} - (\gamma + \alpha + d)Y &= 0 \\ \gamma Y - (\delta + d)Z &= 0 \\ (A - B) - dN - \alpha Y &= 0\end{aligned}\quad (3.3)$$

IV. Main results

**Theorem 4.1.** From the system (3.2) it follows that

- (i) if  $\sigma \leq 1$ , then there is no positive equilibrium;
- (ii) if  $\sigma > 1$ , then there is a unique positive equilibrium  $P_e = (X_e, Y_e, Z_e)$  of the system (3.1), called the “endemic equilibrium”, given by

$$\begin{aligned}
 X_e &= \frac{(\gamma + \alpha + d)(1 + bY_e)}{\beta - a(\gamma + \alpha + d)} \\
 Y_e &= \frac{(\delta + d)[(A - B)\beta - (\gamma + \alpha + d)\{(A - B)a + d\}]}{bd(\gamma + \alpha + d)(\delta + d) + [\alpha(\delta + d) + d(\gamma + \alpha + d)][\beta - a(\gamma + \alpha + d)]} \\
 Z_e &= \frac{\gamma Y_e}{\delta + d} \\
 N_e &= \frac{(A - B) - \alpha Y_e}{d}
 \end{aligned}
 \tag{4.1}$$

It is clear that the limit set of system (3.1) is on the plane  $X + Y + Z = \frac{A - B}{d}$ . Thus we focus on the reduced system

$$\begin{aligned}
 \frac{dY}{dt} &= \frac{d\beta Y}{(d + aA) + (b - a)dY - adZ} \left( \frac{A - B}{d} - Y - Z \right) - (\gamma + \alpha + d)Y \equiv P(Y, Z) \\
 \frac{dZ}{dt} &= \gamma Y - (\delta + d)Z \equiv Q(Y, Z)
 \end{aligned}
 \tag{4.2}$$

**Theorem 4.2.** System (4.2) does not have nontrivial periodic orbits if  $(2d + \gamma + \delta + \alpha)(b - a) > a\gamma$ .

**Proof.** Since  $Y > 0$  and  $Z > 0$ . Take a Dulac function

$$D(Y, Z) = \frac{\{d + a(A - B)\} + (b - a)dY - adZ}{d\beta Y}$$

We have

$$\frac{\partial(DP)}{\partial Y} + \frac{\partial(DQ)}{\partial Z} = -1 - \frac{(\delta + d)\{d + a(A - B)\}}{d\beta Y} - [(2d + \gamma + \delta + \alpha)(b - a) - a\gamma] < 0$$

if  $(2d + \gamma + \delta + \alpha)(b - a) > a\gamma$

In order to study the properties of the disease-free equilibrium  $P_0$  and the endemic equilibrium  $P_e$ .

**Theorem 4.3.** The equilibrium  $P_0 = \left( \frac{A - B}{d}, 0, 0 \right)$  is locally asymptotically stable if  $\sigma \leq 1$  and  $P_0$  is saddle point if  $\sigma > 1$ .

**Proof.** The Jacobian of system (3.1) at  $P_0$  is

$$J(P_0) = \begin{pmatrix} -d & -\frac{\beta(A - B)}{d + a(A - B)} & \delta \\ 0 & \frac{\beta(A - B)}{d + a(A - B)} - (\gamma + \alpha + d) & 0 \\ 0 & \gamma & -(\delta + d) \end{pmatrix}$$

The characteristic equation is

$$(d + t)(\delta + d + t) \left[ \frac{\beta(A - B)}{d + a(A - B)} - (\gamma + \alpha + d) - t \right] = 0 \tag{4.3}$$

The roots of (4.3) are

$$-d, \quad -(\delta + d) \text{ and } \frac{\beta(A - B)}{d + a(A - B)} - (\gamma + \alpha + d)$$

The first two roots having negative real parts and third root  $\frac{\beta(A - B)}{d + a(A - B)} - (\gamma + \alpha + d)$  will have negative real part if  $\sigma \leq 1$ . Thus all roots of (4.3) have negative real parts so  $P_o$  is locally asymptotically stable if  $\sigma \leq 1$  and the root  $\frac{\beta(A - B)}{d + a(A - B)} - (\gamma + \alpha + d)$  will have positive real part if  $\sigma > 1$  so  $P_o$  is saddle point.

**Theorem 4.4.** The equilibrium  $P_o = \left( \frac{A - B}{d}, 0, 0 \right)$  is globally asymptotically stable if  $\sigma \leq 1$ .

**Proof.** Since the set  $T = \left\{ (X, Y, Z) : X + Y + Z \leq \frac{A - B}{d} \right\}$  is attractive and positive invariant.

To prove that all paths in  $T$  approach  $P_o = \left( \frac{A - B}{d}, 0, 0 \right)$  for  $\sigma \leq 1$ , define the Liapunov function  $L = Y$  in  $T$  with

$$\frac{dL}{dt} = \frac{dY}{dt} = \left[ \frac{\beta X}{1 + aX + bY} - (\gamma + \alpha + d) \right] Y \leq 0. \tag{4.4}$$

The Lasalle-Liapunov theory [15] implies that all paths in  $T$  approach the largest positively invariant subset of the set  $T$  where  $\frac{dL}{dt} = 0$ .

Here  $\frac{dL}{dt} = 0$  only if  $Y = 0$  or  $(X, Y, Z) = P_o$ . The positively invariant subset of the plane  $Y = 0$  is the point  $P_o$  so  $P_o$  is globally asymptotically stable for  $\sigma \leq 1$ . To study the properties of the endemic equilibrium  $P_e$ . Let us define

$$x = \frac{\beta}{\delta + d} Y, \quad y = \frac{\beta}{\delta + d} Z, \quad \tau = (\delta + d)t$$

We obtain

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{px}{1 + qx - ry} (K - x - y) - mx, \\ \frac{dy}{d\tau} &= sx - y, \end{aligned} \tag{4.5}$$

Where

$$\begin{aligned} p &= \frac{d}{d + a(A - B)}, \quad q = \frac{(\delta + d)d(b - a)}{\beta\{d + a(A - B)\}}, \quad r = \frac{a(\delta + d)d}{\beta\{d + a(A - B)\}}, \\ K &= \frac{(A - B)\beta}{d(\delta + d)}, \quad m = \frac{\gamma + \alpha + d}{\delta + d}, \quad s = \frac{\gamma}{\delta + d}. \end{aligned}$$

For equilibrium point set,

$$\frac{dx}{d\tau} = 0 \text{ and } \frac{dy}{d\tau} = 0$$

We obtain, two equilibrium point  $(0, 0)$  and  $(x_e, y_e)$  where

$$x_e = \frac{Kp - m}{p(1 + s) + m(q - rs)}, \quad y_e = sx_e$$

The trivial solution  $(0, 0)$  of system (4.5) is the disease-free equilibrium  $P_o$  of model (3.1) and the unique positive equilibrium  $(x_e, y_e)$  of system (4.5) is the endemic equilibrium  $P_e$  of model (3.1) if and only if  $Kp - m > 0$  and  $q - rs > 0$ .

**Theorem 4.5.** Suppose  $m - Kp < 0$ , then there is a unique endemic equilibrium  $(x_e, y_e)$  of model (4.5) which is a stable node.

**Proof.** The Jacobian of system (4.5) at  $(x_e, y_e)$  is

$$J = \begin{pmatrix} \frac{px_e[sx_e(r + q) - (1 + Kq)]}{(1 + qx_e - rsx_e)^2} & \frac{px_e[(Kq - 1) - x_e(r + q)]}{(1 + qx_e - rsx_e)^2} \\ s & -1 \end{pmatrix}$$

$$\det J = \frac{px_e[(1 + s) + K(q - rs)]}{(1 + qx_e - rsx_e)^2}$$

Since  $q > rs, \det(J) > 0$  when  $m - Kp < 0$  and

$$tr(J) = \frac{[ps(r + q)x_e - p(1 + Kq)]x_e - [x_e(rs - q) - 1]^2}{(1 + qx_e - rsx_e)^2}$$

The sign of  $tr(J)$  is determined by

$$S = [ps(r + q)x_e - p(1 + Kq)]x_e. \text{ Substituting } x_e = \frac{Kp - m}{p(1 + s) + m(q - rs)} \text{ into } S, \text{ We have}$$

$$S = \frac{p[-K(p + mq)(q - rs) - (mqs + mq + p + ps)](Kp - m)}{[p(1 + s) + m(q - rs)]^2}.$$

Since  $q > rs, [p(1 + s) + m(q - rs)]^2 > 0$  and,  $[-K(p + mq)(q - rs) - (mqs + mq + p + ps)] < 0$

hence  $S < 0$  if  $m - Kp < 0$ . However, when  $m - Kp < 0$ , we have  $tr(J) < 0$ .

This completes the proof.

**Theorem 4.6.** The equilibrium  $P_e = (X_e, Y_e, Z_e)$  is globally asymptotically stable if  $\sigma > 1$ .

Proof. The proof can be obtained by theorem 4.5.

### V. Example

In this section, we give an example to demonstrate the results obtained in the previous sections.

**Example 5.1.** We take the parameters of the system as  $d = 2.37, a = 3.5, b = 3, A = 6.5, B = 3, \delta = 1.2, \alpha = 0.19, \beta = 10, \gamma = 0.20$ . Then  $P_o = (1.4768, 0, 0)$  and  $\sigma = 0.1819 < 1$ . Therefore, by theorem 4.4,  $P_o$  is a global asymptotically stable in the first octant.

Now we take the parameter of the system as  $d = 0.37, a = 3.5,$

$b = 3, A = 6.5, B = 3, \delta = 1.2, \alpha = 0.19, \beta = 5, \gamma = 0.20.$

Then  $P_e = (4.8307, 4.6245, 0.5891)$  and  $\sigma = 29.1252 > 1$ . Therefore, by theorem 4.6,  $P_e$  is a global asymptotically stable in the interior of the first octant.

## VI. Conclusion

In this paper, we have considered the rich dynamics SIRS epidemic model with an asymptotically homogeneous transmission function, disease related death rate and emigration rate. We have carried out the global qualitative analysis of a realistic SIRS model. Our main results shows that when  $\sigma \leq 1$ , the disease-free equilibrium  $P_0$  is globally asymptotically stable. When  $\sigma > 1$ , the endemic equilibrium  $P_e = (X_e, Y_e, Z_e)$  exists and is globally asymptotically stable.

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