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Numerical Integration and a Proposed Rule

Md.Amanat Ullah
(Mathematics, Uttara University, Bangladesh)

ABSTRACT : *Numerical integration plays very important role in Mathematics. There are a large number of numerical integration methods in the literature and this paper overviews on the most common one, namely the Quadrature method including the Trapezoidal, Simpson's and Weddle's rule. Different procedures are compared and tried to evaluate the more accurate values of some definite integrals. Then it is sought whether a particular method is suitable for all cases. A combined approach of different integral rules has been proposed for a definite integral to get more accurate value for all cases.*

Keywords -*Integration, Quadrature formula, Trapezoidal rule, Simpson's $\frac{1}{3}$ rule, Weddle's rule.*

I. INTRODUCTION

Numerical integration is the study of how the approximate numerical value of a definite integral can be found. It is helpful for the following cases:

- Many integrals can't be evaluated analytically or don't possess a closed form solution. For example: $\int_0^t e^{-x^2}\,dx$.
- Closed form solution exists, but numerical evaluation of the answer can be bothersome.
- The integrand $f(x)$ is not known explicitly, but a set of data points is given for this integrand.
- The integrand $f(x)$ may be known only at certain points, such as obtained by sampling.

Numerical integration of a function of a single variable is called Quadrature, which represents the area under the curve $f(x)$ bounded by the ordinates x_0, x_n and x-axis. The numerical integration of a multiple integral is sometimes described as Cubature.

Numerical integration problems go back at least to Greek antiquity when e.g. the area of a circle was obtained by successively increasing the number of sides of an inscribed polygon. In the seventeenth century, the invention of calculus originated a new development of the subject leading to the basic numerical integration rules. In the following centuries, the field became more sophisticated and, with the introduction of computers in the recent past, many classical and new algorithms had been implemented leading to very fast and accurate results.

An extensive research work has already been done by many researchers in the field of numerical integration. M. Concepcion Ausin[1] compared different numerical integration producers and discussed about more advanced numerical integration procedures. Gordon K. Smith[2] gave an analytic analysis on numerical integration and provided a reference list of 33 articles and books dealing with that topic. Rajesh Kumar Sinha[3] worked to evaluate an integrable polynomial discarding Taylor Series. Gerry Sozio[4] analyzed a detailed summary of various techniques of numerical integration. J. Oliver[5] discussed the various processes of evaluation of definite integrals using higher-order formulae. Otherwise, every numerical analysis book contains a chapter on numerical integration. The formulae of numerical integrations are described in the books of S.S. Sastry[6], R.L. Burden[7], J.H. Mathews[8] and many other authors. In this paper, a Quadrature formula has been used to get the different rules of numerical integrations.

For a given set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of a function $y = f(x)$, which is not known explicitly, if is required to replace $f(x)$ by an interpolating polynomial $\varphi(x)$. Different integration formula can be obtained depending upon the type of the interpolating formula used. The general formula for numerical integration using Newton's forward difference formula is

$\int_{x_0}^{x_n} y\ dx = hn[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n^2-4n+4)}{12}\Delta^3 y_0 + \ \dots\ + (n+1)^{th}\ term] \dots\ \dots$ (i),

where$h$ is the width of the subinterval and n is the number of subintervals.

For n = 1, the equation (i) reduces to $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}[y_0 + y_1]$

Repeating the process for the next intervals and combining all, we get

$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + \dots \dots + y_{n-1}) + y_n] \dots \dots \dots$ (ii), which is known as composite Trapezoidal rule or simply Trapezoidal rule[6].

When n = 2, the equation (i) yields to $\int_{x_0}^{x_2} y\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$

Continuing the process for the next intervals and combining them, we get

$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \dots \dots + y_{n-1}) + 2(y_2 + y_4 + \dots \dots + y_{n-2}) + y_n] \dots \dots$ (iii),

which is known as composite Simpson's $\frac{1}{3}$ rule or simply Simpson's $\frac{1}{3}$ rule[6].

For n = 3 and n = 4, Simpson's $\frac{3}{8}$ and Boole's rules are obtained respectively but Simpson's $\frac{3}{8}$ rule is not as accurate as Simpson's $\frac{1}{3}$rule [6].

For n = 6, the equation (i) yields to

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6].$$

Repeating the process for the next intervals and combining all, we get

$\int_{x_0}^{x_n} y\,dx = \frac{h}{10}[y_0 + 5(y_1 + y_5 + y_7 + \dots \dots + y_{n-1}) + (y_2 + y_4 + y_8 + \dots \dots + y_{n-2}) + 6(y_3 + y_9 + \dots \dots + y_{n-3}) + 2(y_6 + y_{12} + \dots \dots + y_{n-6}) + y_n]$ ………… (iv),

which is known as Weddle's rule.

Here a comparison among Trapezoidal, Simpson's $\frac{1}{3}$ and Weddle's rule is shown in the following table to examine the better rule for accuracy.

Integral	Exact Value	Trapezoidal	Simpson's $\frac{1}{3}$	Weddle's
$\int_0^1 \sqrt{1-x^2}dx$	0.78539816	0.77834373	0.78262639	0.78311087
	Error	0.00705443	0.00277178	0.00228730
$\int_0^2 (e^{x^2} - 1)dx$	14.45262777	14.93311330	14.47143621	14.45523911
	Error	0.48048553	0.01880844	0.00261135
$\int_{0.1}^{2.5} (3logx + 2x^2)dx$	10.77895602	10.72289932	10.76522243	10.77098228
	Error	0.05605670	0.01373358	0.00797374
$\int_0^2 \sqrt{1+3sin^2x}\,dx$	3.26107456	3.25966472	3.26108019	3.26106689
	Error	0.00140984	0.00000563	0.00000768
$\int_0^1 (1 + e^{-x}\cos(4x))\,dx$	1.00745963	1.00882686	1.00749796	1.00746025
	Error	0.00136722	0.00003833	0.00000062

Table-1: A comparison among Trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Weddle's rule for n = 12

From the table-1, it is concluded that the Weddle's rule is more accurate among them and then the Simpson's $\frac{1}{3}$ rule, but we know that for the Weddle's rule and Simpson's $\frac{1}{3}$ rule, the number of subintervals must be divisible by 6 and 2 respectively. But in some real situation, it will may not be found the number of subintervals that is divisible by 6 or 2. For example, if the limits of an integral is 0 to 1, then to use Weddle's rule, the length of subinterval will be taken approximately 0.166667, 0.083333, 0.0555556, … etc. involving errors. If it is chosen an exact length of the subinterval like as 0.2, 0.1, 0.05, …, then the number of subintervals will be 5, 10, 20,… respectively. In those cases, which rule will be used? Can it be used the Weddle's or Simpson's $\frac{1}{3}$ rule?

Suppose the number of subintervals is an odd number and then the same examples are shown in table-2 and table-3 for Weddle's rule and Simpson's $\frac{1}{3}$ rule respectively.

Integral	Exact value	For n = 11	For n = 13	For n = 15	For n = 17
$\int_0^1 \sqrt{1-x^2}dx$	0.78539816	0.76926180	0.77437914	0.77271621	0.77699034
	Error	0.01613636	0.01101902	0.01268195	0.00840782
$\int_0^2 (e^{x^2}-1)dx$	14.45262777	11.06287216	12.11401731	11.29416797	11.9212543
	Error	3.38975562	2.33861046	3.15845980	2.53137347
$\int_{0.1}^{2.5} (3logx+2x^2)dx$	10.77895602	9.21202478	9.72263745	9.39714488	9.74478881
	Error	1.56693124	1.05631857	1.38181114	1.03416721
$\int_0^2 \sqrt{1+3sin^2x}\,dx$	3.26107456	3.08729737	3.14344553	3.10857069	3.14961252
	Error	0.17377719	0.11762903	0.15250387	0.11146204
$\int_0^1 (1+e^{-x}\cos(4x))\,dx$	1.00745963	0.97520534	0.98568941	0.97889657	0.98606461
	Error	0.03225429	0.02177022	0.02856306	0.02139503

Table-2: Weddle's rule for n = 11, n = 13, n = 15 and n = 1

Integral	Exact value	For n = 11	For n = 13	For n = 15	For n = 17
$\int_0^1 \sqrt{1-x^2}dx$	0.78539816	0.77879864	0.78029230	0.78127118	0.78198025
	Error	0.00659952	0.00510586	0.00412698	0.00341791
$\int_0^2 (e^{x^2}-1)dx$	14.45262777	14.78689790	14.66206601	14.59780633	14.55619259
	Error	0.33427013	0.20943824	0.14517856	0.10356482
$\int_{0.1}^{2.5} (3logx+2x^2)dx$	10.77895602	10.77157016	10.77456156	10.77621633	10.77718833
	Error	0.00738586	0.00439446	0.00273969	0.00176769
$\int_0^2 \sqrt{1+3sin^2x}\,dx$	3.26107456	3.26039821	3.26066659	3.26081369	3.26089730
	Error	0.00067635	0.00040797	0.00026087	0.00017726
$\int_0^1 (1+e^{-x}\cos(4x))\,dx$	1.00745963	1.00760821	1.00755069	1.00751499	1.00749533
	Error	0.00014858	0.00009106	0.00005536	0.00003570

Table-3: Simpson's $\frac{1}{3}$ rule for n = 11, n = 13, n = 15 and n = 17

Comparing the values of integral (from Table-1, Table-2 and Table-3) for different number of subintervals, It has been seen that the Weddle's rule and the Simpson's $\frac{1}{3}$ rule can not be used when the number of subintervals is not divisible by 6 and 2 respectively. In this situation, a new method has been proposed.

II. PROPOSED RULE

Let the number of subintervals is n in which first $n_1$is divisible by 6.So Weddle's rule is applicable for the first n_1 subintervals. From the remaining (n-n_1) subintervals, let next n_2 subintervals are divisible by 2.Then Simpson's$\frac{1}{3}$ rule can be used for these subinterval (from n_1-n_2). Trapezoidal rule can be used for the last subintervals if n is an odd number. For example: If n=13, then Weddle's rule can be used for the first 12 subintervals and Trapezoidal rule can be used for the last subinterval. If n=14, then the Weddle's rule can be used for the first 12 subintervals and Simpson's$\frac{1}{3}$ rule can be used for the last two subintervals. If n=15, then Weddle's rule can be used for the first 12 subintervals, Simpson's $\frac{1}{3}$ rule can be used for the next 2 subintervals and Trapezoidal rule can be used for the last subinterval and so on. Finally it is added all the values obtained from these rules to get the values of an integral which are show in table-4.

Integral	Rules	For n=13	For n=14	For n=15	Exact Value
$\int_0^1 \sqrt{1-x^2}\, dx$	Combined	0.78029230	0.78320648	0.78127118	0.78539816
	Trapezoidal	0.779140612	0.77979801	0.78034785	
	Simpson $\frac{1}{3}$	---	0.78319969	---	
$\int_0^2 (e^{x^2}-1)dx$	Combined	14.66206360	14.46073778	14.59780633	14.45262777
	Trapezoidal	14.87970376	14.82130280	14.77408874	
	Simpson's $\frac{1}{3}$	---	14.46301848	---	
$\int_{0.1}^{2.5} (3logx + 2x^2)dx$	Combined	10.77456156	10.77403940	10.77621633	10.77895602
	Trapezoidal	10.73047512	10.73662659	10.74168779	
	Simpson's $\frac{1}{3}$	---	10.77003672	---	
$\int_0^2 \sqrt{1+3sin^2 x}\, dx$	Combined	3.26066659	3.26107446	3.26081369	3.26107456
	Trapezoidal	3.25987344	3.26003901	3.26017256	
	Simpson's $\frac{1}{3}$	---	3.26107741	---	
$\int_0^1 (1+e^{-x}\cos(4x))\, dx$	Combined	1.00755069	1.00744598	1.00751499	1.00745963
	Trapezoidal	1.00862398	1.00846317	1.00833353	
	Simpson's $\frac{1}{3}$	----	1.00744897	---	

Table- 4: Comparison among Combined rule, Trapezoidal rule Simpson's$\frac{1}{3}$ rule for n=13, n=14 and n=15

Table - 4 shows that combined rule gives more accurate result. Beside the above five example, so many examples have been tried and more accurate results are found.

III. CONCLUSION

In this article, to find the a numerical approximate value of a definite integral $\int_{x_0}^{x_n} f(x)dx$, Trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Weddle's rule are used and it is seen that the Weddle's rule gives more accuracy than Simpson's rule. But these rules can't be used for all cases. In those cases it may be used theproposed rule to get the better result.

REFERENCE

[1] M. Concepcion Ausin, 2007, an introduction to quadrature and other numerical integration techniques, Encyclopedia of Statistics in Quality and reliability. Chichester, England.

[2] Gordon K. Smith, 2004, Numerical Integration, Encyclopedia of Biostatistics.2nd edition, Vol-6

[3] Rajesh Kumar Sinha, Rakesh Kumar ,2010, Numerical method for evaluating the integrable function on a finite interval, International Journal of Engineering Science and Technology.Vol-2(6)

[4] Gerry Sozio, 2009, Numerical Integration, Australian Senior Mathematics Journal, Vol-23(1)

[5] J. Oliver, 1971, The evaluation of definite integrals using high-order formulae, The Computer Journal, Vol-14(3)

[6] S.S Sastry, 2007, Introductory Method of Numerical Analysis, Fourth Edition, Prentice-hall of India Private Limited.

[7] Richard L. Burden, 2007, Numerical Analysis, Seven Edition, International Thomson Publishing Company.

[8] Jonh H. Mathew, 2000, Numerical Method for Mathematics, Science and Engineering, Second Edition, Prentice Hall of India Private Limited.