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Existence of Solution of Fractional Impulsive Delay Integrodifferential Equation in Banach Space

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Abstract. *This paper mainly concerned with existence of solution of fractional delay integrodifferential equation in Banach space. The results are obtained by the fixed point theorem.*

1. Introduction

Fractional calculus deals with generalization of integrals and derivatives of noninteger order. Fractional calculus involves a wide area of applications by bringing into a broader paradigm conceps of physics, mathematics and engineering [12, 14]. In [3, 13] the authors have provided the existence of solutions of abstract fractional differential equations by using fixed point techniques. Several authors have been discussed different types of nonlinear fractional differential and integrodifferential equations in Banach spaces.

The theory of impulsive differential equations has undergone rapid developments over the years and played a very important role in modern applied mathematical modeling of real processes arising in phenomena studied in physics, population dynamics, chemical technology and economics. In [2, 8] Benchohra et al. established sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < q \leq 1$ and $1 < q \leq 2$.

Anguraj and Karthikeyan [4] proved existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators. Benchohra and Seba [7] studied the existence of nonlocal Cauchy problem for semilinear fractional evolution equations.Balchandran and Trujjilo [5] investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces.

In the paper we consider the fractional impulsive delay integrodifferential equation of the form

(1.1)

$$\begin{cases} {}^cD_t^q(x(t)) = A(t,x)x(t) + f(t, x_t, \int_0^t h(t,s,x_s)ds), & t \in J = [0,T], \quad t \neq t_k \\ \Delta x|_{t=t_{t_k}} = I_k(x(t_k^-)), & t = t_k, \quad k = 1,2,...,n. \\ x(0) = x_0 \end{cases}$$

where $0 < q < 1$ and the state $x(.)$ belongs to Banach space X endowed with the $\|.\|$, $A(t,x)$ is a bounded linear operator on a Banach space X. D_t^q is the Caputo fractional derivative. f is a continuous function. $I_k : X \to X$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h\to 0^+} x(t_k+h)$, $x(t_k^-) = \lim_{h\to 0^-}(t_k + h)$, $k = 1,2,3,....n$, $0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$.

The rest of this paper is organized as follows. In section 2, some preliminaries are presented. In section 3, we study the existence and uniqueness of solutions for the impulsive fractional system (1.1). In section 4, an example is given.

2. Preliminaries

In this section we introduce some basic definitions, notations, lemmas and mathematical preliminaries which are used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integrable operator of order $q \geq 0$ of function $f \in L^1(R^+)$ is defined as

$$I_{0^+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad t > 0$$

where $\Gamma(.)$ is the Euler gamma function and L^σ is Banach space of Lebesgue measurable functions with $\|l\|_{L^\sigma} > \infty$.

Next we introduce the Caputo fractional derivative.

Definition 2.2. The Caputo fractional derivative of order $q \geq 0$, $n - 1 < q < n$, is defined as

$$D_{0^+}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{(n-q-1)} f^{(n)}(s)ds, \quad t > 0$$

where the function $f(t)$ has absolutely continuous derivative up to order

$(n-1)$.
If $0 < q < 1$, then

$$D_{0+}^{q} f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{(-q)} f^{(1)}(s) ds$$

where $f^{(1)}(s)ds = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

Lemma 2.3. *For $q > 0$ the solution of fractional differential equation ${}^cD_t^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}.$$

where $c_i \in R, i = 0, 1, 2, \ldots\ldots, n-1$, $(n = [q]+1)$ where $[q]$ denotes the integer part of $q > 0$.

Lemma 2.4. *A function $x : (-\infty, T] \to X$ is said to be a solution of system (1.1) if $x(0) = x_0$, the impulsive condition $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, 3, \ldots n$ is verified the restriction of $x(.)$, to the interval $J_k(k = 0, 1, 2, \ldots n)$ is continuous and following integral equation holds for $t \in J$. (See [1], Lemma 4.2)*

(2.1)

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x) x(s) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x) x(s) ds \\ + \frac{1}{\Gamma(q)} \sum\limits_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x_s, \int_0^s h(s,\tau,x_\tau) d\tau) ds \\ + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x_s, \int_0^s h(s,\tau,x_\tau) d\tau) ds + \sum\limits_{0<t_k<t} I_k(x(t_k^-)). \end{cases}$$

3. Main result

We assume following conditions to prove existence of the solution of equation (1.1).

(H1) $A : J \times X \to B(X)$ is a continuous bounded linear operator and there exists a constant $M > 0$, such that the function satisfies the Lipschitz condition

$$\|A(t,x) - A(t,y)\|_{B(X)} \le M\|x - y\|, \quad \text{for all} \quad x, y \in X.$$

(H2) $f : J \times X \times X \to X$ is continuous and there exists a constant $L > 0$, such that

$$\|f(t,u,x) - f(t,v,y)\| \le L\|u-v\| + \|x-y\|, \quad \text{for all} \quad x,y,u,v \in X.$$

(H3) $h : \Delta \times X \to X$ is continuous and there exists a constant $L_1 > 0$, such that

$$\|h(t,s,x) - h(t,s,y)\| \le L_1\|x-y\|, \quad \text{for all} \quad x,y \in X.$$

(H4) The functions $I_k : X \to X$ are continuous and there exists a constant $L_2 > 0$, such that

$$\|I_k(x) - I_k(y)\| \le L_2\|x-y\|, \quad \text{for each} \quad x,y \in X \quad \text{and} \quad k = 1,2,...m.$$

Let $B_r = \{u \in X : \|u\| \le r\}$ for some $r > 0$. For bravity let us take $\gamma = \frac{T^q}{\Gamma(q+1)}$ and $K = \sup_{t\in J}\|A(t,0)\|$, $N = \max_{t\in J}\|f(t,0,0)\|$, $N_1 = \max_{(t,s)\in\Delta}\|h(t,s,0)\|$.

From (H1) we observe that for any $x \in B_r$,

$$\|A(t,x)\| \le \|A(t,x) - A(t,0)\| + \|A(t,0)\| \le M\|u\| + \|A(t,0)\| \le Mr + K$$

Further we assume that

(H5) $\|x_0\| + \gamma(m+1)((M_r + K)r + M_0) + mL_2r \le r$ where $M_0 = Lr + LL_1Tr + LN_1T + N$.

(H6) Let $p = \gamma(m+1)[(2Mr+K+L+LL_1T)+mL_2]$ be such that $0 \le p < 1$.

Theorem 3.1. *If the hypothesis (H1)-(H6) are satisfied, then fractional delay integrodifferential equation (1.1) has a unique solution in J.*

Proof. Let $\Omega = PC(J : B_r)$ as in [6]. Define the mapping $\Phi : \Omega \to \Omega$ by

(3.1)

$$\begin{aligned}\Phi x(t) =& x_0 + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s,x)x(s)ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s,x)x(s)ds \\ &+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds \\ &+ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t_k-s)^{q-1}f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds + \sum_{0<t_k<t} I_k(x(t_k^-)).\end{aligned}$$

and we have to show that Φ has a fixed point. This fixed point is then a solution of equation (1.1). We know from [6] that $\Phi B_r \subset B_r$. From the assumptions we have

$$\|\Phi x(t)\| \leq \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s,x)\| \|x(s)\| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s,x)\| \|x(s)\| ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau)\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau)\| ds + \sum_{0<t_k<t} \|I_k(x(t_k^-))\| \leq r.$$

Thus Φ maps B_r into itself. Now for $x, x' \in \Omega$, we have

$$\|\Phi x(t) - \Phi x'(t)\| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s,x)x(s) - A(s,x')x'(s)\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s,x)x(s) - A(s,x')x'(s)\| ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau) - f(s, x'_s, \int_0^s h(s,\tau,x'_\tau)d\tau)\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau) - f(s, x'_s, \int_0^s h(s,\tau,x'_\tau)d\tau)\| ds$$
$$+ \sum_{0<t_k<t} \|I_k(x(t_k^-)) - I_k(x'(t_k^-))\|$$
$$\leq [\gamma(m+1)(2Mr + K + L + LL_1T) + mL_2] \|x(t) - x'(t)\|$$
$$\leq p\|x - x'\|.$$

Since $0 \leq p < 1$, then Φ is a contraction mapping and therefore there exists a unique fixed point $x \in \Omega$ such that $\Phi x(t) = x(t)$. Any fixed point of Φ is the solution of (1.1). □

4. Example

Consider the following fractional delay integrodifferential equation with impulsive condition of the form

(4.1)
$$^cD^q x(t) = \frac{1}{24}(1 + \sin x(t))x(t) + \frac{1}{(x+3)^2} \frac{x(\sin t)}{1 + x(\sin t)} + \frac{1}{9} \int_0^t e^{-\frac{1}{4}x(\sin s)} ds, \quad t \in J, \quad t \neq \frac{1}{2},$$
(4.2)
$$\Delta x|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2}^-)|}{3 + |x(\frac{1}{2}^-)|},$$
(4.3)
$$x(0) = x_0.$$

where $0 < q \leq 1$, take $J = [0,1]$, $T = 1$

Let

$$A(t,x) = \frac{1}{24}(1+\sin x(t)),$$

$$\text{let} \quad H(x_t) = \int_0^t h(t,s,x_s)ds = \int_0^t e^{-\frac{1}{4}x(\sin s)}ds$$

$$f(t,x,H(x_t)) = \frac{1}{(t+3)^2}\frac{x(\sin t)}{1+x(\sin t)} + H(x_t), \quad t \in J, \quad x \in X.$$

Let $u, v \in X$ and $t \in J$. Then we have

$$\|H(x_t)-H(y_t)\| = \left\|\int_0^t e^{-\frac{1}{4}x(\sin s)}ds - \int_0^t e^{-\frac{1}{4}y(\sin s)}ds\right\| \le \frac{1}{4}\|x(\sin t)-y(\sin t)\| \le \frac{1}{4}\|x-y\|.$$

$$\begin{aligned}\|f(t,x,H(x_t)) - f(t,y,H(y_t))\| &= \left\|\frac{1}{(t+3)^2}\left(\frac{x(\sin t)}{1+x(\sin t)} - \frac{y(\sin t)}{1+y(\sin t)}\right) + \frac{1}{9}(H(x_t)-H(y_t))\right\| \\ &\le \left\|\frac{1}{(t+3)^2}\left(\frac{x(\sin t)}{1+x(\sin t)} - \frac{y(\sin t)}{1+y(\sin t)}\right)\right\| + \left\|\frac{1}{9}(H(x_t)-H(y_t))\right\| \\ &\le \frac{1}{(t+3)^2}\|x-y\| + \frac{1}{9}\|H(x_t)-H(y_t)\| \le \frac{1}{9}(\|x-y\| + \|H(x_t)-H(y_t)\|.\end{aligned}$$

Hence the condition (H1)-(H3) hold with $M = \frac{1}{24}$, $L = \frac{1}{9}$, $L_1 = \frac{1}{4}$.Here $K = \frac{1}{24}$. Let $x, y \in X$, we have by (H4)

$$\|I_k(x) - I_k(y)\| = \left\|\frac{x}{3+x} - \frac{y}{3+y}\right\| = \frac{3\|x-y\|}{(3+x)(3+y)} \le \frac{1}{3}\|x-y\|.$$

Note that $L_2 = \frac{1}{3}$. Choose $r = 1$, $m = 1$. We shall check the condition

$$\gamma(m+1)(2Mr + K + L + LL_1T) + mL_2 < 1$$

It satisfies indeed

$$\gamma(m+1)(2Mr + K + L + LL_1T) + mL_2 < 1 \Leftrightarrow \Gamma(q+1) > \frac{19}{24}. \tag{4.4}$$

which is satisfied for some $q \in (0,1]$. Further (H5) is satisfied by a suitable choice of x_0. Then by theorem 3.1 the problem (4.1)-(4.3) has a unique solution on [0,1] for the values of q satisfying (4.4).

5. Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] R. P. Agarwal, M. Benchohra ,S. Hamani , A survey on the existence results for boundary value problems of nonlinear equations and inclusions, Acta Applicandae Mathematicae. 109, (2010),973-1033.

[2] R. P. Agarwal, M. Benchohra, B. A. Slimani, Existence results for differential equations with fractional order and impulses, Memoire on Differential Equations and Mathematical Physics. 44, (2008),1-21.

[3] A. Anguraj, P. Karthikeyan, G. M. N'Guereata, Nonlocal Cauchy problem for some fractional abstract integrodifferential equations in Banach spaces, Communications in Mathematical Analysis. 6(1), (2009),31-35.

[4] A. Anguraj, P. Karthikeyan, Y. K. Chang, Existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators, Nonlinear Analysis: Theory Methods and Applications. 71, (2009),4377-4386.

[5] K. Balchandran, JJ. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces, Nonlinear Analysis: Theory Methods and Applications. 72, (2010),4587-4593.

[6] K. Balchandran, S. Kiruthika, JJ. Trujillo, Existence results for fractional integrodifferential equations in Banach spaces, Commun. Non- linear Sci Numer Simulat. 16 (4), (2011),1970-1977.

[7] M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach spaces, Electronic Journal of Qualitative Theory of Differential Equations Special Edition. 8, (2009),1-14.

[8] M. Benchohra, B. A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electronic Journal of Differential Equations. 10, (2009),1-11.

[9] B. Bonilla, M. Rivero, L. Rodriguez-Germa, JJ. Trujillo ,Fractional differential equations as alternative models to nonlinear differential equations, Applied Mathematics and Computation. 187, (2007),79-88.

[10] Y. K. Chang, Controllability of impulsive fractional differential systems with infinite delay in Banach spaces, Chaos, Solitons and Fractals. 33, (2007),1601-1609.

[11] Giesle M Mophou, Existence and uniqueness of solutions to impulsive fractional differential equations, Nonlinear analysis. 72, (2010),1604- 1615.

[12] JH. He, Approximate analytical solution for seepay flow with fractional derivative in porous media, Computer Methods in Applied Mechanics and Engineering. 167, (1998),57-68.

[13] E. Hernandez, D. O'Regan, K. Balchandran, On recent developments in the theory of abstract differential equations with fractional derivatives, Nonlinear Analysis: Theory, Methods and Applications. 73(10), (2010),3462-3471.

[14] R. Hilfer, Applications of fractional calculus in physics, Singapore; World Scientific;2000.