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Common Fixed Point Theorems for Hybrid Pairs of Occasionally Weakly Compatible Mappings in b-Metric Space.

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ABSTRACT: *The objective of this paper is to obtain some common fixed point theorems for hybrid pairs of single and multi-valued occasionally weakly compatible mappings in b-metric space.*

KEYWORDS: *Occasionally weakly compatible mappings, single and multi—valued maps, common fixed point theorem, b- metric space.*
2000 Mathematics Subject Classification: 47H10; 54H25.

I. INTRODUCTION

The study of fixed point theorems, involving four single-valued maps, began with the assumption that all of the maps are commuted. Sessa [8] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [5] and then pairwise weakly compatible maps [6]. Jungck and Rhoades [7] introduced the concept of occasionally weakly compatible maps.

Abbas and Rhoades [1] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weakly compatible (owc).
The concept of $b-metric\ space$ was introduced by Czerwik[3]. Several papers deal with fixed point theory for single and multi- valued maps in $b-metric\ space$.
In this paper we extend the result of Hakima Bouhadjera [2] from $metric\ space$ to $b-metric\ space$.

II. PRELIMINARY NOTES

Let (X,d) denotes a metric space and $CB(X)$ the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on $CB(X)$ induced by the metric d; i.e.,
$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\}$
for A,B in $CB(X)$, where

$$d(x,A) = \inf\{d(x,y): y \in A\}.$$

Definition2.1.[3] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d: X\times X \longrightarrow R_+$ (nonnegative real numbers) is called a $b-metric$ provided that, for all $x,y,z \in X$,
(bi) $d(x,y) = 0\ iff\ x = y$,
(bii) $d(x,y) = d(y,x)$,
$(biii)$ $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair (X,d) is called $b-metric\ space$ with parameter s.
It is clear that the definition of $b-metric\ space$ is an extension of usual metric space. Also, if we consider $s = 1$ in above definition, then we obtain definition of usual metric space.

Definition2.2.[1] Maps $f: X \longrightarrow X$ and $T: X \longrightarrow CB(X)$ are said to be occasionally weakly compatible (owc) if and only if there exist some point x in X such that $fx \in Tx$ and $fTx \subseteq Tfx$.
For our main results we need the following lemma. We cite the following lemma from Czerwik [3,4].

Lemma2.3. Let (X,d) be any $b-metric\ space$ and let $A,B \in CB(X)$, then for any $a \in A$ we have

$$d(a,B) \le H(A,B).$$

III. MAIN RESULTS

Theorem3.1 Let (X,d) be a b-metric space with parameter $s \ge 1$. Let $f,g:X \longrightarrow X$ and $F,G:X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc and satisfy inequality

$$H(Fx,Gy) \le \lambda[max\left\{d(fx,gy),d(fx,Fx),d(gy,Gy),\frac{1}{2}[d(fx,Gy)+d(gy,Fx)]\right\}] \qquad (3.1)$$

for all $x,y \in X$, where $s\lambda \in [0,\frac{1}{2})$. Then f,g,F & G have a unique common fixed point in X.

Proof: Since the pairs $\{f,F\}$ and $\{g,G\}$ are owc, then there exist two elements $u,v \in X$ such that $fu \in Fu, fFu \subseteq Ffu$ and $gv \in Gv, gGv \subseteq Ggv$.

First we prove that $fu = gv$. By Lemma [2.3] and by $(biii)$ we have $d(fu,gv) \le s\,H(Fu,Gv)$. Suppose that $H(Fu,Gv) > 0$. Then by (3.1) we get

$$H(Fu,Gv) \le \lambda[max\left\{d(fu,gv),d(fu,Fu),d(gv,Gv),\frac{1}{2}[d(fu,Gv)+d(gv,Fu)]\right\}]$$

Since $d(fu,Gv) \le H(Fu,Gv)$ and $d(gv,Fu) \le H(Fu,Gv)$ by Lemma [2.3], and then

$$H(Fu,Gv) \le \lambda \max\{sH(Fu,Gv),H(Fu,Gv)\} = s\lambda\, H(Fu,Gv)$$

This inequality is false as $s\lambda \in \left[0,\frac{1}{2}\right)$, unless $H(Fu,Gv) = 0$ which implies that $fu = gv$.

Again by Lemma [2.3] and $(biii)$ we have

$d(f^2u,fu) = d\big(f(fu),g(v)\big) \le s\,H(Ffu,Gv)$. We claim that $f^2u = fu$. Suppose not. Then $H(Ffu,Gv) > 0$ and using inequality (3.1) we get

$$H(Ffu,Gv) \le \lambda[max\left\{d(ffu,gv),d(ffu,Ffu),d(gv,Gv),\frac{1}{2}[d(ffu,Gv)+d(gv,Ffu)]\right\}].$$

But $d(f^2u,Gv) \le H(Ffu,Gv)$ and $d(gv,Ffv) \le H(Ffu,Gv)$ by Lemma [2.3] and so

$$H(Ffu,Gv) \le s\lambda\, H(Ffu,Gv),$$

which is false as $s\lambda \in \left[0,\frac{1}{2}\right)$, unless $H(Ffu,Gv) = 0$, thus $f^2u = fu = gv$.

Similarly, we can prove that $g^2v = gv$.

Putting $fu = gv = z$, then $fz = z = gz$, $z \in Fz$ and $z \in Gz$. Therefore z is the common fixed point of maps f,g,F & G.

Now suppose that f,g,F & G have another common fixed point $z \ne z'$. Then by lemma [2.3] and $(biii)$ we have $d(z,z') = d(fz,gz') \le s\,H(Fz,Gz')$.

Assume that $H(Fz,Gz') > 0$. Then the use of (3.1) gives

$$H(Fz,Gz') \le \lambda[max\left\{d(fz,gz'),d(fz,Fz),d(gz',Gz'),\frac{1}{2}[d(fz,Gz')+d(gz',Fz)]\right\}].$$

Since $d(fz,Gz') \le H(Fz,Gz')$ and $d(gz',Fz) \le H(Fz,Gz')$,

we have $H(Fz,Gz') \le s\lambda\, H(Fz,Gz')$.

which is false as $s\lambda \in \left[0,\frac{1}{2}\right)$. Then $H(Fz,Gz') = 0$ and hence $z = z'$.

Corollary3.2 Let (X,d) be a b-metric space with parameter $s \ge 1$. Let $f,g:X \longrightarrow X$ and $F,G:X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc and satisfy inequality

$$H(Fx,Gy) \le \lambda[max\{d(fx,gy),d(fx,Fx),d(fx,Gy),d(gy,Gy),d(gy,Fx)]\}] \qquad (3.2)$$

for all $x,y \in X$, where $s\lambda \in [0,\frac{1}{2})$. Then f,g,F & G have a unique common fixed point in X.

Proof: Clearly the result immediately follows from Theorem 3.1.

Theorem3.3 Let (X,d) be a b-metric space with parameter $s \ge 1$. Let $f,g:X \longrightarrow X$ and $F,G:X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc and satisfy inequality

$$H(Fx,Gy) \le \lambda[\alpha d(fx,gy) + \beta \max\{d(fx,Fx),d(gy,Gy)\} + \gamma \max\{d(fx,gy),d(fx,Gy),d(gy,Fx)\}] \qquad (3.3)$$

for all $x,y \in X$, with $\alpha,\beta,\gamma > 0$ & $(\alpha+\beta+\gamma) = 1$, also $s\lambda \in \left[0,\frac{1}{2}\right)$. Then f,g,F & G have a unique common fixed point in X.

Proof: Since the pairs $\{f,F\}$ and $\{g,G\}$ are owc, then there exist two elements $u,v\in X$ such that $fu\in Fu, fFu\subseteq Ffu$ and $gv\in Gv, gGv\subseteq Ggv$.

First we prove that $fu=gv$. By Lemma [2.3] and by $(biii)$ we have $d(fu,gv)\le s\,H(Fu,Gv)$. Suppose that $H(Fu,Gv)>0$. Then by (3.3) we get

$$H(Fu,Gv)\le\lambda[\alpha d(fu,gv)+\beta\max\{d(fu,Fu),d(gv,Gv)\}+\gamma\max\{d(fu,gv),d(fu,Gv),d(gv,Fu)\}]$$
$$=\lambda[\alpha d(fu,gv)+\gamma\max\{d(fu,gv),d(fu,Gv),d(gv,Fu)]$$

Since $d(fu,Gv)\le H(Fu,Gv)$ and $d(gv,Fu)\le H(Fu,Gv)$ by Lemma [2.3], and then

$$H(Fu,Gv)\le\lambda[\alpha sH(Fu,Gv)+\gamma\max\{sH(Fu,Gv),H(Fu,Gv),H(Fu,Gv)]$$
$$=\lambda[\alpha s\,H(Fu,Gv)+\gamma s\,H(Fu,Gv)]$$
$$=\lambda[(\alpha+\gamma)sH(Fu,Gv)]$$
$$< s\lambda\,H(Fu,Gv),\qquad\text{as }(\alpha+\beta+\gamma)=1.$$

This inequality is false as $s\lambda\in\left[0,\frac{1}{2}\right)$, unless $H(Fu,Gv)=0$ which implies that $fu=gv$.

Again by Lemma [2.3] and $(biii)$ we have

$d(f^2u,fu)=d(f(fu),g(v))\le s\,H(Ffu,Gv)$. We claim that $f^2u=fu$. Suppose not. Then $H(Ffu,Gv)>0$ and using inequality (3.3) we get

$$H(Ffu,Gv)\le\lambda[\alpha d(ffu,gv)+\beta\max\{d(ffu,Ffu),d(gv,Gv)\}$$
$$+\gamma\max\{d(ffu,gv),d(ffu,Gv),d(gv,Ffu)\}]$$
$$=\lambda[\alpha d(ffu,gv)+\gamma\max\{d(ffu,gv),d(ffu,Gv),d(gv,Ffu)]$$

But $d(f^2u,Gv)\le H(Ffu,Gv)$ and $d(gv,Ffv)\le H(Ffu,Gv)$ by Lemma [2.3] and so

$$H(Ffu,Gv)\le\lambda[\alpha sH(Ffu,Gv)+\gamma\max\{sH(Ffu,Gv),H(Ffu,Gv),H(Ffu,Gv)]$$
$$=\lambda[\alpha s\,H(Ffu,Gv)+\gamma s\,H(Ffu,Gv)]$$
$$=\lambda[(\alpha+\gamma)s\,H(Ffu,Gv)]$$
$$< s\lambda\,H(Ffu,Gv),\qquad\text{as }(\alpha+\beta+\gamma)=1.$$

$H(Ffu,Gv)\le s\lambda\,H(Ffu,Gv)$,

which is false as $s\lambda\in\left[0,\frac{1}{2}\right)$, unless $H(Ffu,Gv)=0$, thus $f^2u=fu=gv$.

Similarly, we can prove that $g^2v=gv$.

Putting $fu=gv=z$, then $fz=z=gz$, $z\in Fz$ and $z\in Gz$. Therefore z is the common fixed point of maps $f,g\ F\ \&\ G$.

Now suppose that $f,g,F\ \&\ G$ have another common fixed point $z\neq z'$. Then by lemma and $(biii)$ we have

$d(z,z')=d(fz,gz')\le s\,H(Fz,Gz')$.

Assume that $H(Fz,Gz')>0$. Then the use of (3.3) gives

$H(Fz,Gv)\le\lambda[\alpha d(fz,gz')+\beta\max\{d(fz,Fz),d(gz',Gz')\}+\gamma\max\{d(fz,gz'),d(fz,Gz'),d(gz',Fz)\}]$.

Since $d(fz,Gz')\le H(Fz,Gz')$ and $d(gz',Fz)\le H(Fz,Gz')$,

we have $H(Fz,Gz')< s\lambda\,H(Fz,Gz')$.

which is false as $s\lambda\in\left[0,\frac{1}{2}\right)$. Then $H(Fz,Gz')=0$ and hence $z=z'$.

Theorem3.4 Let (X,d) be a b-metric space with parameter $s\ge 1$. Let $f,g:X\longrightarrow X$ and $F,G:X\longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc and satisfy inequality

$$H^p(Fx,Gy)\le\lambda\left[\alpha d^p(fx,gy)+(1-\alpha)\max\{d^p(fx,gy),d^p(fx,Fx),d^p(gy,Gy),d^{\frac{p}{2}}(gy,Fx).d^{\frac{p}{2}}(fx,Gy)\}\right]$$

(3.4)

for all $x,y\in X$, with $\alpha\in[0,1]$ also $s\lambda\in\left[0,\frac{1}{2}\right)$ and $p\ge 1$. Then $f,g,F\ \&\ G$ have a unique common fixed point in X.

Proof: Since the pairs $\{f,F\}$ and $\{g,G\}$ are owc, then there exist two elements $u,v\in X$ such that $fu\in Fu, fFu\subseteq Ffu$ and $gv\in Gv, gGv\subseteq Ggv$.

First we prove that $fu = gv$. By Lemma [2.3] and by $(biii)$ we have $d(fu, gv) \leq s\, H(Fu, Gv)$. Suppose that $H(Fu, Gv) > 0$. Then by (3.4) we get

$$H^p(Fu, Gv) \leq \lambda\left[\alpha d^p(fu, gv) + (1-\alpha)\max\{d^p(fu, gv), d^p(fu, Fu), d^p(gv, Gv), d^{\frac{p}{2}}(gv, Fu).\, d^{\frac{p}{2}}(fu, Gv)\}\right]$$

Since $d(fu, Gv) \leq H(Fu, Gv)$ and $d(gv, Fu) \leq H(Fu, Gv)$ by Lemma [2.3], and then

$H^p(Fu, Gv) \leq s\lambda\, H^p(Fu, Gv)$

This inequality is false as $s\lambda \in \left[0, \frac{1}{2}\right)$, unless $H(Fu, Gv) = 0$ which implies that $fu = gv$.

Again by Lemma [2.3] and $(biii)$ we have

$d(f^2u, fu) = d\big(f(fu), g(v)\big) \leq s\, H(Ffu, Gv)$. We claim that $f^2u = fu$. Suppose not. Then $H(Ffu, Gv) > 0$ and using inequality (3.4) we get

$$H^p(Ffu, Gv) \leq \lambda\left[\alpha d^p(ffu, gv) + (1-\alpha)\max\{d^p(ffu, gv), d^p(ffu, Ffu), d^p(gv, Gv), d^{\frac{p}{2}}(gv, Ffu).\, d^{\frac{p}{2}}(ffu, Gv)\}\right].$$

But $d(f^2u, Gv) \leq H(Ffu, Gv)$ and $d(gv, Ffv) \leq H(Ffu, Gv)$ by Lemma [2.3] and so

$H^p(Ffu, Gv) \leq s\lambda\, H^p(Ffu, Gv)$,

which is false as $s\lambda \in \left[0, \frac{1}{2}\right)$, unless $H(Ffu, Gv) = 0$, thus $f^2u = fu = gv$.

Similarly, we can prove that $g^2v = gv$.

Putting $fu = gv = z$, then $fz = z = gz$, $z \in Fz$ and $z \in Gz$. Therefore z is the common fixed point of maps f, g, F & G.

Now suppose that f, g, F & G have another common fixed point $z \neq z'$. Then by lemma [2.3] and $(biii)$ we have $d(z, z') = d(fz, gz') \leq s\, H(Fz, Gz')$.

Assume that $H(Fz, Gz') > 0$. Then the use of (3.4) gives

$$H^p(Fz, Gz') \leq \lambda\left[\alpha d^p(fz, gz') + (1-\alpha)\max\{d^p(fz, gz'), d^p(fz, Fz), d^p(gz', Gz'), d^{\frac{p}{2}}(gz', Fz).\, d^{\frac{p}{2}}(fz, Gz')\}\right].$$

Since $d(fz, Gz') \leq H(Fz, Gz')$ and $d\,(gz', Fz) \leq H(Fz, Gz')$,

we have $H^p(Fz, Gz') \leq s\lambda\, H^p(Fz, Gz')$.

which is false as $s\lambda \in \left[0, \frac{1}{2}\right)$. Then $H(Fz, Gz') = 0$ and hence $z = z'$.

If we put in above Theorem $f = g$ and $F = G$, we obtain the following result.

Corollary3.5 Let (X, d) be a b-metric space with parameter $s \geq 1$. Let $f: X \longrightarrow X$ and $F: X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f, F\}$ are owc and satisfy inequality

$$H^p(Fx, Fy) \leq \lambda\left[\alpha d^p(fx, fy) + (1-\alpha)\max\{d^p(fx, fy), d^p(fx, Fx), d^p(fy, Fy), d^{\frac{p}{2}}(fy, Fx).\, d^{\frac{p}{2}}(fx, Fy)\}\right] \tag{3.5}$$

for all $x, y \in X$, with $\alpha \in [0,1]$ also $s\lambda \in \left[0, \frac{1}{2}\right)$ and $p \geq 1$. Then f & F have a unique common fixed point in X.

Now, letting $f = g$ we get the next corollary.

Corollary3.6 Let (X, d) be a b-metric space with parameter $s \geq 1$. Let $f: X \longrightarrow X$ and $F, G: X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f, F\}$ and $\{f, G\}$ are owc and satisfy inequality

$$H^p(Fx, Gy) \leq \lambda\left[\alpha d^p(fx, fy) + (1-\alpha)\max\{d^p(fx, fy), d^p(fx, Fx), d^p(fy, Gy), d^{\frac{p}{2}}(fy, Fx).\, d^{\frac{p}{2}}(fx, Gy)\}\right] \tag{3.6}$$

for all $x, y \in X$, with $\alpha \in [0,1]$ also $s\lambda \in \left[0, \frac{1}{2}\right)$ and $p \geq 1$. Then f, g, F & G have a unique common fixed point in X.

Corollary3.7 Let (X, d) be a b-metric space with parameter $s \geq 1$. Let $f, g: X \longrightarrow X$ and $F, G: X \longrightarrow CB(X)$ be single and multi-valued maps, respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality

$H(Fx,Gy) \leq d(fx,gy) + (\lambda - 1)\max\{d(fx,gy), d(fx,Fx), d(gy,Gy), d(gy,Fx), d(fx,Gy)\}$

(3.7)

for all $x, y \in X$, where $s\lambda \in [0, \frac{1}{2})$. Then f, g, F & G have a unique common fixed point in X.

Proof: Clearly the result immediately follows from Theorem 3.1.

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