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An Implicit Method for Solving Fuzzy Partial Differential Equation with Nonlocal Boundary Conditions

B. Orouji[1], N. Parandin[2] L. Abasabadi[3], A. Hosseinpour[4]
[1]*Department of Mathematics, College of Science, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.*
[2]*Department of Mathematics, College of Science, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.*
[3]*Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran*
[4] *Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran*

Abstract: - In this paper we introduce a numerical solution for the fuzzy heat equation with nonlocal boundary conditions. The main purpose is finding a difference scheme for the one dimensional heat equation with nonlocal boundary conditions. In these types of problems, an integral equation is appeared in the boundary conditions. We first express the necessary materials and definitions, and then consider our difference scheme and next the integrals in the boundary equations are approximated by the composite trapezoid rule. In the final part, we present an example for checking the numerical results. In this example we obtain the Hausdorff distance between exact solution and approximate solution.

Keywords: - *Fuzzy numbers, Fuzzy heat equation, Finite difference scheme, stability*.

I. INTRODUCTION

This paper is concerned with the numerical solution of the heat equation

$$(D_t - a^2 D_x^2)\widetilde{U} = \widetilde{O} \qquad x \in (0,1),\ \ t \in (0,1] \tag{1}$$

Subject to the nonlocal boundary conditions

$$\begin{cases} \widetilde{U}(0,t) = \int_0^1 k_0(x)\widetilde{U}(x,t)dx + \widetilde{g_0}(t) \\ \widetilde{U}(1,t) = \int_0^1 k_1(x)\widetilde{U}(x,t)dx + \widetilde{g_1}(t) \end{cases} \tag{2}$$

And the initial condition

$$\widetilde{U}(x,0) = \tilde{g}(x) \qquad x \in (0,1) \tag{3}$$

Where $\tilde{f}, \tilde{k}_0, \tilde{k}_1, \tilde{g}_0, \tilde{g}_1$ *and* $\tilde{g}$ are known fuzzy functions. Over the last few years, many other physical phenomena were formulated into nonlocal mathematical models [1]. Hence, the numerical solution of parabolic partial differential equations with nonlocal boundary specifications is currently an active area of research. The topics of numerical methods for solving fuzzy differential equations have been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by Chang and Zadeh in [10]. It was following up by Dubois and Prade in [1], who defined and used the extension principle. Other methods have been discussed by Puri and Relescu in [4] and Goestschel and Voxman in [9]. The initial value problem for first order fuzzy differential equations has been studied by several authors [5, 6, 7, 8, and 11]. On the metric space (E^n, D) of normal fuzzy convex sets with the distance D gave by the maximum of the Hausdorff distances between the corresponding levels sets.

II. MATERIALS AND DEFINITIONS

We begin this section with defining the notation we will use in the paper. Let X be a location of objects denoted generically by x, and then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)|x \in X\}$. $\mu_{\tilde{A}}$ is called the membership function or grade of membership of x in $\tilde{A}$. The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite.

Definition 2.1. The set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called α-cut set:

$$A_a = \{x \in X|\mu_{\tilde{A}}(x) \geq \alpha\}$$

$A_a^{'} = \{x \in X|\mu_{\tilde{A}}(x) \geq \alpha\}$ is called strong α-cut.

Definition 2.2. The triangular fuzzy number $\tilde{N}$ is defined by three numbers $\alpha < m < \beta$ as $\tilde{A} = (\alpha, m, \beta)$. This representation is interpreted as membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-\alpha}{m-\alpha} & \alpha \leq x \leq m \\ 1 & x = m \\ \dfrac{x-\beta}{m-\beta} & m < x \leq \beta \\ 0 & o.\omega \end{cases}$$

If $\alpha > 0 (\alpha \geq 0)$ then $\tilde{A} > 0 (\tilde{A} \geq 0)$,
If $\beta < 0 (\beta \leq 0)$ then $\tilde{A} < 0 (\tilde{A} \leq 0)$.

Definition 2.3. An arbitrary number is showed by an ordered pair of functions $\left(\underline{a}(r), \overline{a}(r)\right), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{a}(r)$ is a bounded left semi continuous non-decreasing function over [0,1],
2. $\overline{a}(r)$ is a bounded left semi continuous non-decreasing function over [0,1],
3. $\underline{a}(r) \leq \overline{a}(r)$ $, 0 \leq r \leq 1$.

In particular, if $\underline{a}, \overline{a}$ are linear functions we have a triangular fuzzy number.
A crisp number a is simply represented by $\underline{a}(r) = \overline{a}(r) = \mathrm{a}$ $, 0 \leq r \leq 1$.

Definition 2.4. For arbitrary fuzzy numbers $(\underline{u}(r), \overline{u}(r))$ $, v = (\underline{u}(r), \overline{u}(r))$ we have algebraic operations bellow:

1. $ku = \begin{cases} (k\underline{u}, k\,\overline{u}) & k \geq 0 \\ (k\overline{u}, k\underline{u}) & k < 0 \end{cases}$
2. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$
3. $u - v = (\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r))$
4. $u.v = (mins, maxs)$, which $s = \{\underline{uv}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{uv}\}$.

Remark. Since the α-cut of fuzzy numbers is always a closed and bounded interval, so we can write $\widetilde{A_\alpha} = [\underline{a}(\alpha), \overline{a}(\alpha)]$, for all α.

Definition 2.5. Assume $u = \left(\underline{u}(r), \overline{u}(r)\right)$, $v = (\underline{v}(r), \overline{v}(r))$ are two fuzzy numbers. The Hausdorff metric D_H is defined by:

$$D_H(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\} \quad (4)$$

This metric is a bound for error. By it we obtain the difference between exact solution and approximate solution.

III. FINITE DIFFERENCE METHOD

In this section we solve the fuzzy heat equation by an implicit method. Assume $\widetilde{U}$ is a fuzzy function of the independent crisp variable x and t. We define:

$$I = \{(x, t)|0 \leq x \leq 1, 0 \leq t \leq T\}$$

α-cut of $\widetilde{U}(x, t)$ and it's the parametric form, will be:

$$\widetilde{U}(x, t)[\alpha] = [\underline{U}(x, t; \alpha), \overline{U}(x, t; \alpha)].$$

We let that the $\underline{U}(x, t; \alpha), \overline{U}(x, t; \alpha)$ have continuous partial differential, therefore $(D_t - a^2 D_x^2)\overline{U}(x, t; \alpha)$, and $(D_t - a^2 D_x^2)\underline{U}(x, t; \alpha)$ are continuous for all $(x, t) \in I$, all $\alpha \in [0,1]$. we divide the domain $[0,1] \times [0, T]$ in to $M \times N$ mesh with spatial step size $h = \frac{1}{N}$ in x −direction and in x −direction and $k = \frac{T}{M}$ in t −direction. The gride points are given by:

$$x_i = ih \qquad i = 0,1, \dots, N$$
$$t_j = jk \qquad j = 0,1, \dots, M$$

Denote the value of $\widetilde{U}$ at the representative mesh point $p(x_i, t_j)$ by:

$\widetilde{U}_p = \widetilde{U}(x_i, t_j) = \widetilde{U}_{i,j}$

And also parameter form of fuzzy number $\widetilde{U}_{i,j}$ is:

$$\widetilde{U}_{i,j} = (\underline{U}_{i,j}, \overline{U}_{i,j})$$

We have:

$$\begin{cases} (D_t)\widetilde{U}_{i,j} = (D_t\underline{U}_{i,j}, D_t\overline{U}_{i,j}) \\ (D_x^2)\widetilde{U}_{i,j} = (D_x^2\underline{U}_{i,j}, D_x^2\overline{U}_{i,j}) \end{cases}$$

Then by Taylor's expansion we obtain:

$$\begin{cases} D_x^2\underline{U}_{i,j} \simeq \dfrac{\underline{u}_{i-1,j+1} - 2\overline{u}_{i,j+1} + \underline{u}_{i+1,j+1}}{h^2} \\ D_x^2\overline{U}_{i,j} \simeq \dfrac{\overline{u}_{i-1,j+1} - 2\underline{u}_{i,j+1} + \overline{u}_{i+1,j+1}}{h^2} \end{cases}$$

And also for $(D_t)\widetilde{U}$ at p, we have:

$$\begin{cases} D_t\underline{U}_{i,j} \simeq \frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k} \\ D_t\overline{U}_{i,j} \simeq \frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k} \end{cases} \tag{6}$$

Parametric form of heat equation will be:

$$\begin{cases} D_t\underline{U}_{i,j} - a^2 D_x^2\overline{U}_{i,j} = \tilde{0} \\ D_t\overline{U}_{i,j} - a^2 D_x^2\underline{U}_{i,j} = \tilde{0} \end{cases} \tag{7}$$

By (4) and (5) the difference scheme for heat equation is:

$$\begin{cases} \frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k} - a^2 \frac{\overline{u}_{i-1,j+1} - 2\underline{u}_{i,j+1} + \overline{u}_{i+1,j+1}}{h^2} = 0 \\ \frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k} - a^2 \frac{\underline{u}_{i-1,j+1} - 2\overline{u}_{i,j+1} + \underline{u}_{i+1,j+1}}{h^2} = 0 \end{cases} \tag{8}$$

By above equations we obtain:

$$\begin{cases} -r\overline{u}_{i-1,j+1} + (1+2r)\underline{u}_{i,j+1} - r\overline{u}_{i+1,j+1} = \underline{u}_{i,j} \\ -r\underline{u}_{i-1,j+1} + (1+2r)\overline{u}_{i,j+1} - r\underline{u}_{i+1,j+1} = \overline{u}_{i,j} \end{cases} \tag{9}$$

Where:

$$r = \frac{ka^2}{h^2}$$

$\widetilde{U} = (\underline{u}, \overline{u})$ is the exact solution of the approximating difference equations, and $x_i, (i = 1, \ldots, N-1)$ and $t_j, (j = 0,1, \ldots, M)$.

We have $2(N-1)$ equations with $2(N+1)$ unknowns. Therefore we need other four equations. We obtain these equations by boundary conditions (2) are described by the trapezoid rule. So

$$a_0\widetilde{U}_{0,j+1} + \sum_{i=1}^{N-1} a_i\widetilde{U}_{i,j+1} + a_N\widetilde{U}_{i,j+1} \approx -\tilde{g}_{0,i+1}$$

$$b_0\widetilde{U}_{0,j+1} + \sum_{i=1}^{N-1} b_i\widetilde{U}_{i,j+1} + b_N\widetilde{U}_{i,j+1} \approx -\tilde{g}_{1,i+1}$$

Where

$$a_0 = \frac{h}{2}k_0(x_0) - 1 \qquad a_N = \frac{h}{2}k_0(x_N)$$

$$b_N = \frac{h}{2}k_1(x_N) - 1 \qquad b_0 = \frac{h}{2}k_1(x_0)$$

And

$$a_i = hk_0(x_i) \quad , \quad b_i = hk_1(x_i) \qquad i = 1, \ldots, N-1$$

Also parametric form of fuzzy numbers $\tilde{g}_0$ and $\tilde{g}_1$ are:

$$\tilde{g}_0 = \left(\underline{g}_0, \overline{g}_0\right) \qquad \tilde{g}_1 = \left(\underline{g}_1, \overline{g}_1\right)$$

By equations (9) we obtain:

$$-r\widetilde{U}_{i-1,j+1} + (1+2r)\widetilde{U}_{i,j+1} - r\widetilde{U}_{i+1,j+1} = \widetilde{U}_{i,j} \qquad \begin{matrix} i = 1, \ldots, N-1 \\ j = 0,1, \ldots, M \end{matrix}$$

Therefore equations can be written in matrix form as:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_N \\ -r & 1+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 1+2r & -r \\ b_0 & \ldots & b_{N-2} & b_{N-1} & b_N \end{pmatrix}$$

Then we will have:

$$A\widetilde{U}_{j+1} = \widetilde{U}_j$$

The coefficients matrix of this system i.e. $A = (a_{ij})$ is a crisp matrix $(N+1) \times (N+1)$, and $\widetilde{U}_{j+1} = (\tilde{u}_{1,j+1}, \ldots, \tilde{u}_{N,j+1})^T$, $\widetilde{U}_j = (\tilde{u}_{1j}, \ldots, \tilde{u}_{Nj})^T$ are fuzzy vectors in the parametric form. Where $\tilde{u}_{1,j+1} = (\underline{u}_{i,j+1}, \overline{u}_{i,j+1})$ and $\tilde{u}_{ij} = (\underline{u}_{ij}, \overline{u}_{ij})$. So we have to solve a system of order $2(N+1) \times 2(N+1)$. We re-arrangement this linear system of equations as follows:

$$SX = Y \qquad (10)$$

where

$$X = (\underline{u}_{0,j+1}, \ldots, \underline{u}_{N,j+1}, -\overline{u}_{0,j+1}, \ldots, -\overline{u}_{N,j+1})^T$$
$$Y = (\underline{u}_{0,j}, \ldots, \underline{u}_{N,j}, -\overline{u}_{0,j}, \ldots, -\overline{u}_{N,j})^T$$

And the matrix S is defined as follows:

$$a_{ij} \geq 0 \;\Rightarrow\; s_{ij} = s_{i+N+1,j+N+1} = a_{ij}$$
$$a_{ij} < 0 \;\Rightarrow\; s_{i,j+N+1} = s_{i+N+1,j} = -a_{ij}$$

the rest of matrix elementary s_{ij} which do not get these relations are zero.

IV. NUMERICAL EXAMPLE

In this section we present a numerical example to illustrate our method, whose exact solution is known to us. Consider the fuzzy heat equation

$$\frac{\partial \widetilde{U}}{\partial t}(x,t) - \frac{1}{\pi^2}\frac{\partial^2 \widetilde{U}}{\partial x^2}(x,t) = \tilde{0} \qquad 0 < x < 1\,, \qquad t > 0$$

Subject to the nonlocal boundary conditions

$$\widetilde{U}(0,t) = \int_0^1 x\widetilde{U}(x,t)dx + \left(1 + \frac{2}{\pi^2}\right)\exp(-t)$$
$$\widetilde{U}(1,t) = \int_0^1 x\widetilde{U}(x,t)dx - \left(1 - \frac{2}{\pi^2}\right)\exp(-t)$$

and the initial condition

$$\widetilde{U}(x,0) = \widetilde{K}\cos \pi x$$

and $\widetilde{K}[\alpha] = \left[\underline{k}(\alpha), \overline{k}(\alpha)\right] = [\alpha - 1, 1 - \alpha]$. which is easily seen to have exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,t;\alpha) - \frac{1}{\pi^2}\frac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha) = 0 - \alpha$$
$$\frac{\partial \overline{U}}{\partial t}(x,t;\alpha) - \frac{1}{\pi^2}\frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha) = 0 + \alpha$$

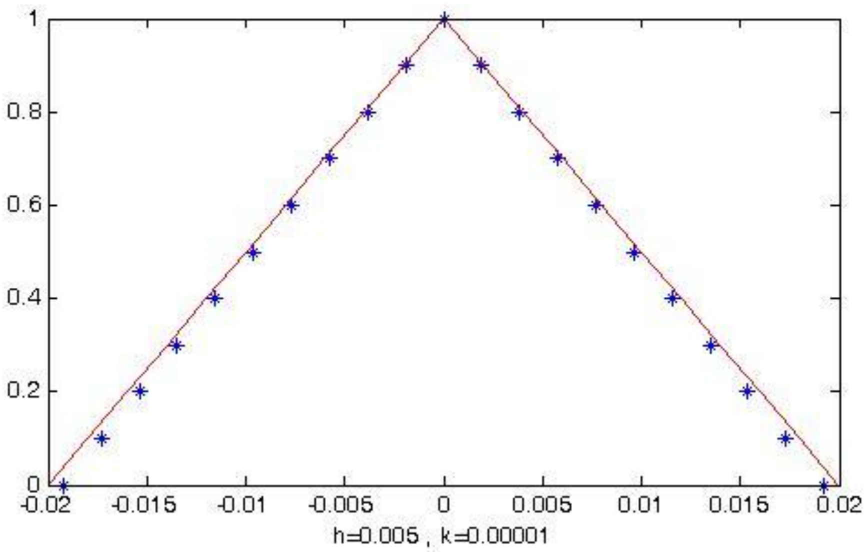
are

$$\underline{U}(x,t;\alpha) = \begin{cases} \underline{k}(\alpha)\exp(-t)\cos \pi x & 0 < x < \frac{1}{2} \\ \overline{k}(\alpha)\exp(-t)\cos \pi x & \frac{1}{2} < x < 1 \end{cases}$$

and

$$\underline{U}(x,t;\alpha) = \begin{cases} \overline{k}(\alpha)\exp(-t)\cos \pi x & 0 < x < \frac{1}{2} \\ \underline{k}(\alpha)\exp(-t)\cos \pi x & \frac{1}{2} < x < 1 \end{cases}$$

The exact and approximate solutions are shown in next figure at the point (0.2,0.001) with $h = 0.005, k = 0.00001$. The housdroff distance between solutions in this case is $7.58e - 004$.



V. CONCLUSION

Our purpose in this article is solving fuzzy partial differential equation (FPDE). We presented an implicit method for solving this equation, and we considered necessary conditions for stability of this method. In last section we given an example for consider numerical results. Also we compared the approximate solution and exact solution. Then we obtained the Hausdorf distance between them in two cases.

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REFERENCES

[1] D. Dubois and H. Prade, Towards fuzzy differential calculus: part 3, differentiation, Fuzzy Sets and Systems, 8 (1982) 225-233.
[2] G. D. Smith, Numerical solution of partial differential equations, (1993).
[3] M. Friedman and M. Ming, A. Kandel, Fuzzy linear systems, FSS, 96 (1998) 201-209.
[4] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Appl, 91 (1983) 321-325.
[5] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24 (1987) 301-307.
[6] O. Kaleva, The cauchy problem for fuzzy differential equations, Fuzzy sets and systems, 5 (1990) 389-396.
[7] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets. World Scientific, Singapore, (1994).
[8] P. E. Kloeden, Remarks on Peano-like theorems for fuzzy differential equations, Fuzzy Sets and Systems, 44 (1991) 161-163.
[9] R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy sets and Systems, 18 (1988) 31-43.
[10] S. L. Chang and L.A. Zadeh, On fuzzy mapping and control, IEEE Trans, Systems Mah Cybemet, 2 (1972) 30-34.
[11] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24 (1987) 319-330.
[12] T. Allahviranloo and N. Ahmadi and E. Ahmadi and Kh. Shams Alketabi, Block Jacobi two-stage method for fuzzy systems of linear equations, Applied Mathematics and Computation, 175 (2006) 1217-1228.