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Remarks on one S.S. Dragomir's Result

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Abstract: - *In theorem* 1 [1], *S.S. Dragomir gave bounds for the normalized Jensen functional defined by convex function* f *, which one is defined on strictly convex subset* C *of vector space* X *. Further, using inequality* (2.1) *of normed space* $(X,\|\cdot\|)$ *he proved the inequalities* (3.1), (3.2) *and* (3.3), *and after that from inequality* (3.3) *he performed inequality* (3.6), *which was previously proved in* [2]. *In this paper we'll give an example, which shows that inequalities* (3.3) *are not correct and will show how the inequality* (3.2) *implies* (3.6) *given in* [1].

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I. INTRODUCTION

Let X be a vector space, C convex subset of X , P_n set of all nonnegative $n-$tuples $(p_1, p_2,..., p_n)$ such that $\sum_{i=1}^{n} p_i = 1$, $f : C \to \mathbf{R}$ a convex function, $\mathbf{x} = (x_1, x_2,..., x_n) \in C$, $\mathbf{p} \in P_n$ and

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \geq 0 \tag{1}$$

be the normalized Jensen functional. In [1], for functional (1), S.S. Dragomir gave elementary proof of the following theorem (theorem of bounds for the normalized Jensen functional).

Theorem 1. If $\mathbf{p}, \mathbf{q} \in P_n$, $q_i > 0$, for each $i = 1,2,...,n$ then

$$J_n(f, \mathbf{x}, \mathbf{q}) \max_{1 \leq i \leq n} \{\frac{p_i}{q_i}\} \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq J_n(f, \mathbf{x}, \mathbf{q}) \min_{1 \leq i \leq n} \{\frac{p_i}{q_i}\} . \blacksquare \tag{2}$$

Furthermore, using the fact, that in normed space $(X,\|\cdot\|)$, the function $f_p : X \to \mathbf{R}$, $f_p(x) = \| x \|^p$, $p \geq 1$ is convex on X , S.S. Dragomir proved that inequality (2) implies the following inequalities

$$\begin{aligned} &[\sum_{j=1}^{n} q_j \| x_j \|^p - \| \sum_{j=1}^{n} q_j x_j \|^p] \max_{1 \leq i \leq n} \{\frac{p_i}{q_i}\} \geq \sum_{j=1}^{n} p_j \| x_j \|^p - \| \sum_{j=1}^{n} p_j x_j \|^p, \\ &\sum_{j=1}^{n} p_j \| x_j \|^p - \| \sum_{j=1}^{n} p_j x_j \|^p \geq [\sum_{j=1}^{n} q_j \| x_j \|^p - \| \sum_{j=1}^{n} q_j x_j \|^p] \min_{1 \leq i \leq n} \{\frac{p_i}{q_i}\}, \end{aligned} \tag{3}$$

And letting $q_j = \frac{1}{n}$, for $j = 1,2,...,n$ he get the following inequalities

$$[\sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\|\sum_{j=1}^{n}x_j\|^p]\max_{1\le i\le n}\{p_i\} \ge \sum_{j=1}^{n}p_j\|x_j\|^p - \|\sum_{j=1}^{n}p_jx_j\|^p,$$
$$\sum_{j=1}^{n}p_j\|x_j\|^p - \|\sum_{j=1}^{n}p_jx_j\|^p \ge [\sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\|\sum_{j=1}^{n}x_j\|^p]\min_{1\le i\le n}\{p_i\}. \tag{4}$$

Finally, letting $p_i = \frac{1}{\|x_i\|}$, for $x_i \in X \setminus \{0\}, i = 1,2,...,n$ and also using the inequalities (4) S.S. Dragomir get the following :

$$[\sum_{j=1}^{n}\|x_j\|^{p-1} - \|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|^p]\max_{1\le i\le n}\{\|x_i\|\} \ge \sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\|\sum_{j=1}^{n}x_j\|^p,$$
$$\sum_{j=1}^{n}\|x_j\|^p - n^{1-p}\|\sum_{j=1}^{n}x_j\|^p \ge [\sum_{j=1}^{n}\|x_j\|^{p-1} - \|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|^p]\min_{1\le i\le n}\{\|x_i\|\}, \tag{5}$$

By which for $p = 1$ he get the following

$$[n - \|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|]\max_{1\le i\le n}\{\|x_i\|\} \ge \sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|,$$
$$\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\| \ge [n - \|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|]\min_{1\le i\le n}\{\|x_i\|\}, \tag{6}$$

proved in [2], and their generalization was given by Mitani, Saito, Kato and Tamura, [3] and also by Pečarić and Rajić, [4].

II. MAIN COMMENT

Example 1. Let $X = \mathbf{R}^n$, $n \ge 2$ and $\|\cdot\|$ be an Euclid's norm. Then the vertex $x_i = (0,...,0,\underset{i}{1},0,...,0)$, $i = 1,2,...,n$ satisfy the following

$$\|x_i\| = 1, \frac{x_i}{\|x_i\|} = x_i, \text{ for } i = 1,2,...,n \text{ and } \|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\| = \|\sum_{i=1}^{n}x_i\| = \sqrt{n}.$$

According to this, for $p > 1$ the inequalities (5) applies the following ones

$$n - n^{\frac{p}{2}} \ge n - n^{1-p}n^{\frac{p}{2}} \ge n - n^{\frac{p}{2}},$$

So, we get that for $n \ge 2$ and $p > 1$ is true that $n^{p-1} = 1$, and that is contradiction. ■

At first, it seemed that inequalities (3) - (6) get correct by (2). So this procedure [2] is citated by L. Maligranda. However, the example 1 shows that inequality (5) is not correct if $p > 1$. The error occurred in a choice of numbers $p_i = \frac{1}{\|x_i\|}$, $x_i \in X \setminus \{0\}$, $i = 1,2,...,n$. In fact, according to Theorem 1 these numbers have to satisfy the condition $\sum_{i=1}^{n} p_i = 1$. The mentioned condition is not satisfied for arbitrary vectors $x_i \in X \setminus \{0\}$, $i = 1,2,...,n$ and for thus selected numbers $p_i, i = 1,2,...,n$

Anyway, Theorem 1, i.e. inequality (4) implies (6).

Let $\alpha_i > 0$, for $i = 1,2,...,n$, and $p_i = \frac{\alpha_i}{\sum_{k=1}^{n}\alpha_k}, i = 1,2,...,n$. So, $\mathbf{p} \in P_n$ and If we take that $p_i = \frac{\alpha_i}{\sum_{k=1}^{n}\alpha_k}, i = 1,2,...,n$ in (4), we get the following inequalities:

$$[\sum_{i=1}^{n}\|x_i\|^p - n^{1-p}\|\sum_{i=1}^{n}x_i\|^p]\max_{1\le i\le n}\{\alpha_i\} \ge \sum_{i=1}^{n}\alpha_i\|x_i\|^p - (\sum_{i=1}^{n}\alpha_i)^{1-p}\|\sum_{i=1}^{n}\alpha_ix_i\|^p,$$
$$\sum_{i=1}^{n}\alpha_i\|x_i\|^p - (\sum_{i=1}^{n}\alpha_i)^{1-p}\|\sum_{i=1}^{n}\alpha_ix_i\|^p \ge [\sum_{i=1}^{n}\|x_i\|^p - n^{1-p}\|\sum_{i=1}^{n}x_i\|^p]\min_{1\le i\le n}\{\alpha_i\}. \tag{7}$$

Letting $\alpha_i = \frac{1}{\|x_i\|}$, for $x_i \in X \setminus \{0\}, i = 1,2,...,n$, in inequalities (7) we get the following:

$$\sum_{i=1}^{n} \| x_i \|^p - n^{1-p} \| \sum_{i=1}^{n} x_i \|^p \geq [\sum_{i=1}^{n} \| x_i \|^{p-1} - (\sum_{i=1}^{n} \frac{1}{\|x_i\|})^{1-p} \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \|^p] \min_{1 \leq i \leq n} \{\| x_i \|\},$$

$$[\sum_{i=1}^{n} \| x_i \|^{p-1} - (\sum_{i=1}^{n} \frac{1}{\|x_i\|})^{1-p} \| \sum_{i=1}^{n} \frac{x_i}{\|x_i\|} \|^p] \max_{1 \leq i \leq n} \{\| x_i \|\} \geq \sum_{i=1}^{n} \| x_i \|^p - n^{1-p} \| \sum_{i=1}^{n} x_i \|^p .$$

Finally, by using the inequalities above for $p = 1$, we get inequalities (6).

Remarks. In the end, we can note that for $\alpha_i = \| x_i \|$, $x_i \in X$, $i = 1,2,...,n$, inequality (7) implies the following inequalities

$$[\sum_{i=1}^{n} \| x_i \|^p - n^{1-p} \| \sum_{i=1}^{n} x_i \|^p] \max_{1 \leq i \leq n} \{\| x_i \|\} \geq \sum_{i=1}^{n} \| x_i \|^{p+1} - (\sum_{i=1}^{n} \| x_i \|)^{1-p} \| \sum_{i=1}^{n} \| x_i \| x_i \|^p ,$$

$$\sum_{i=1}^{n} \| x_i \|^{p+1} - (\sum_{i=1}^{n} \| x_i \|)^{1-p} \| \sum_{i=1}^{n} \| x_i \| x_i \|^p \geq [\sum_{i=1}^{n} \| x_i \|^p - n^{1-p} \| \sum_{i=1}^{n} x_i \|^p] \min_{1 \leq i \leq n} \{\| x_i \|\},$$

In which, for $p = 1$ we get the inequalities below:

$$[\sum_{i=1}^{n} \| x_i \| - \| \sum_{i=1}^{n} x_i \|] \max_{1 \leq i \leq n} \{\| x_i \|\} \geq \sum_{i=1}^{n} \| x_i \|^2 - \| \sum_{i=1}^{n} \| x_i \| x_i \|,$$

$$\sum_{i=1}^{n} \| x_i \|^2 - \| \sum_{i=1}^{n} \| x_i \| x_i \| \geq [\sum_{i=1}^{n} \| x_i \| - \| \sum_{i=1}^{n} x_i \|] \min_{1 \leq i \leq n} \{\| x_i \|\}.$$

Similarly, as (6), if the vertex $x_i \in X$, $i = 1,2,...,n$ are such that $\| x_i \| = 1$, then they become equalities.

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