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Asian Rainbow Put Option Pricing Determination based on Fractional Brownian Motion using Geometric and Arithmetic Averages

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ABSTRACT: *Financial markets that have changed over time have had very diverse effects in the development of various types of complex financial derivatives, especially in the options field. In this paper, the combination of two exotic options, namely rainbow options and Asian options under fractional Brownian motion is discussed.Fractional Brownian motion was chosen because it is better at describing the situations of fluctuations in stock prices change. The Asian rainbow put option pricing formula has been obtained, then the analytical results and simulations are presented in table form. Sensitivity analysis is also carried out to illustrate the rationality of the model used for important parameters, such as correlation coefficient, Hurst index, and the risk-free rate.Next, this paper discussed the comparison of Asian rainbow put option pricing using geometric and arithmetic averages.*

KEYWORDS*: fractional Brownian motion, rainbow put Asian option, geometric average, arithmetic average*

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I. INTRODUCTION

Financial markets are markets that trade financial assets, there are the money markets and the capital market.The capital market consists of the bond market, stock market, and market for derivatives (Hermuningsih[1]).Derivatives are financial instruments which value depends on or will be valued if the price of the underlying asset has changed.Generally, there are two types of derivatives, namely options and futures which are currently actively traded on many financial exchanges because they are considered beneficial for investors who want to play on the stock market and financial world. Options is a contract that gives the holder the right to buy or sell assets (according to the agreement) at a certain time and certain price (Hull [2]).

Call option is a type of contract that gives the option holder the right to buy from the seller of a certain number of shares at a specified price and period, while the put option is an option that gives the holder the right to sell shares in a certain amount to the buyer of the option at a predetermined time and price. There is an option where the payoff does not only depend on the price of the asset at the time of implementation, but also depends on the price of the asset during the option's validity period.This option is calledpath-dependent optionoralso known as an exotic options.There are two main types of exotic options, namely correlation options dan path dependent options. Correlation optionsare options whosetheir payoffis affected by more than oneunderlying asset.One of the example ofcorrelation optionsis Rainbow option. Path dependent optionsare options whose payoff is affected by how the price of the underlying assets at maturity is reached as expected.One of example ofpath dependent optionsis Asian option(Wiklund [3]).

The Asian option is an option where the payoff depends on the average price of the asset as long as the option runs. There are two ways to determine the price of Asian put or call options, which are analytically reduced or numerically approximated.Asian options and Rainbow options are greatly increasing in popularity in the open market.The Asian Rainbow option is the development of options whose the results depend on the maximum or minimum average price of the underlying asset (Wang, et al. [4]).The optio values can be determined using several models, one of them using the Black-Scholes model.The Black-Scholes model can be used for Asian options from the option granted until the option expires.The Black-Scholes model has several assumptions, there are no dividends, no transaction costs, risk-free interest rates, and changes in stock prices following a random pattern (Widyawati and Sulistianingsih[5]).

Stock price changes that occur in the financial markets move randomly according to time.The changes that occur are assumed to follow the Wiener process or commonly called the geometric Brownian motion which is particular type of Markov chain stochastic process, where a change occurs in a short time and the current value is influential to predict future values (Hull [2]). In the financial mathematic, the Black-Scholes stock price model consist of risky assets, $S(t)$ stocks, bonds, and risk-free assets. A risky asset is a stochastic process $S(t)$ that follows geometric Brownian motion. In the Black-Scholes equation model, the return is independent each other, in other words today's stock prices changes have no correlation with changes in previous stock prices (Ostaszewicz, [6]). Several studies from Mandelbrot [7]have proven that a long-range dependency exist between returns on several stock markets, so Mandelbrot [7] suggests the replacement of Brownian motion reduction model with fractional Brownian motion (B_t^H).

Fractional Brownian motionis an extension of classical or simple Brownian motion with the addition of Hurst index H, where when the Hurst parameter$H = 1/2$is a standard geometric Brownian motion (Wang [4]). In Peters [8]stated if the time series of a stock price has a high Hurst parameter, then the stock price will be less risky and the noise in the data will also be less. For this reason, the application of fractional Brownian motion is submitted with the aim of minimizing risk in the stock price model.The Hurst parameters between$0 \leq H \leq 1$intervals are divided into three different groups. If the Hurst parameter$H = 1/2$then the stock prices changes followrandom walksor standard geometric Brownian motion, the return are not correlated and random. If$0 \leq H \leq 1/2$then the stock price movement has anti-persistent behavior which the mean is experiencing reverting.If the Hurst parameter is between$1/2 \leq H \leq 1$then stock prices changes have a persistent behavior, namelytrend reinforcing(Mandelbrot and Hudson [9]).

The previous studies that focused on making fractional Brownian motion as a basis for determining option prices were conducted by Elliot and Chan [10]whi examined Perpetual American options with their stock price movements following fractional Brownian motion with the Hurst parameter $H \in (0,1)$. In their research also explained when the Hurst parameter$H = 1/2$, then the stock price has the same result as the Merton type. Rostek and Schöbel [11] who discuss direct arguments in mathematics that clarify when and why fractional Brownian motion is appropriate for use in economics modeling. Next, Stulz [12] presents the formula of call and put option prices from European options when the maximum or minimum of two risky assets using the Black-Scholes equation model whose stock price movements follow geometric Brownian motion. Bin and Fei [13] in their research discussed the Asian Rainbow call option pricing formula with two underlying assets whose stock price movements followed geometric Brownian motion and then they compared their average Asian stock using the arithmetic and geometric average. Bin dan Fei are also looking for a parity relationship between the Asian rainbow call and put options. Next, Wang et al. [4] do the research about the Asian Rainbow call option pricing formula with two underlying asset using the Black-Scholes model under the fractional Brownian motion with Hurst parameters$H > 1/2$.

Based on some of the previous studies above, this research will examine the Asian Rainbow put option pricing which is a combination of the Rainbow option and the Asian option for two assets using geometric and arithmetic averages for the average Asian stock. Then, the analytical of the Asian Rainbow put option pricing formula using geometric average is performed using the partial differential equation (PDE) approach, which is a fractional Brownian motion that is solved using Wick integral or Ito integral. Therefore, the Black-Scholes formula under fractional Brownian motion must be satisfy the basic assumption of no arbitrage, self-similarity, and long-range dependence. Furthermore, to ensure the accuracy of the Asian Rainbow put option pricing formula where the average for Asian option using geometric and arithmetic averages will be compared the analytical solutions and numerical solutions using Monte Carlo simulations.

II. ASIAN RAINBOW OPTION PRICING FORMULA USING GEOMETRIC AVERAGE

The stock price equation model can be expressed in a differential equation involving a random variable based on time, commonly called stochastic calculus.The model that can be used is the Brownian motion stock price model that contain stochastic process. Under the assumption of continous time, the model for the two types of assets S_1 and S_2 satisfies the following differential equation:

$$\begin{cases} dS_{1t} = \mu_1 S_{1t} dt + \sigma_1 S_{1t} dB \\ dS_{2t} = \mu_2 S_{2t} dt + \sigma_2 S_{2t} dB \end{cases} \tag{1}$$

where$S$is a stock price, $\mu$is themean or constant drift, $\sigma$is the volatility or uncertainty factor, anddB is a stochastic process that is standard or geometric Brownian motion. In this case the fractional Brownian motion is used which involves the Hurst index in the equation model to capture (find) fluctuations in stock prices, so the equation (1) becomes:

$$\begin{cases} dS_{1t} = \mu_1 S_{1t} dt + \sigma_1 S_{1t} dB_{1t}^H \\ dS_{2t} = \mu_2 S_{2t} dt + \sigma_2 S_{2t} dB_{2t}^H \end{cases} \tag{2}$$

where the two assets are correlated, so the correlation coeeficient between dB_{1t}^H and dB_{2t}^H is ρ. In this study used fixed strike price put option where the geometric average price of option price $G(t)$ is satisfied by $G(t) = \exp\left(\frac{1}{T}\int_0^t \ln S(t)\, dt\right)$, so the option price function form (V) forthe geometric Asian rainbow put options of two different assets at time t specified by the underlying asset S_1 and S_2 at time t, the variables I_1 and I_2, and thestrike price K are

$$V(S_1, S_2, I_1, I_2, t) = \max\big(K - \big(G_1(t), G_2(t)\big), 0\big) \tag{3}$$

the path dependent option values nilai-nilai are on I_1 and I_2, but basically it's still formed by the underlying assets in the equation (1). Based on the model in equation (2), for example $V(\cdot)$ is the option price which depends on S_1 and S_2 of the Asian Rainbow option at time t, by applying Ito's Lemma which was expanded using the Taylor series with the Hurst parameter $H \in [0,1]$, then you will get the equation:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}(dS_2)^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2}dS_1 dS_2 + a(t). \tag{4}$$

The correlation coeffiient between dB_{1t}^H dan dB_{2t}^H is ρ, then dB_{1t}^H and dB_{2t}^H can be transformed into two fBm which are interdependent from equation (1)so that they are obtained:

$$(dS_{it})^2 = \mu_i^2 S_i^2 (dt)^2 + 2\mu_i S_i^2 \sigma_i dt\, dB_{it}^H + \sigma_i^2 S_i^2 (dB_{it}^H)^2,$$

with $(dB_{it}^H)^2 = 2Ht^{2H-1}$ and $b(dt) = \mu_i^2 S_i^2 (dt)^2 + 2\mu_i S_i^2 \sigma_i dt\, dB_{it}^H$ so:

$$(dS_{it})^2 = 2H\sigma_i^2 S_{it}^2 t^{2H-1} dt + b(dt). \qquad (i = 1,2) \tag{5}$$

Next, $dS_{1t} dS_{2t}$ will be presented as follows:

$$dS_{1t}dS_{2t} = S_{1t}S_{2t}(\mu_1\mu_2(dt)^2 + \mu_1\sigma_2 dt\, dB_{2t}^H + \mu_2\sigma_1 dt\, dB_{1t}^H) + \sigma_1\sigma_2 S_{1t}S_{2t} dB_{1t}^H dB_{2t}^H,$$

with $dB_{1t}^H dB_{2t}^H = 2\rho\sigma_1\sigma_2 S_{1t}S_{2t}Ht^{2H-1}dt$ and $c(t) = S_{1t}S_{2t}(\mu_1\mu_2(dt)^2 + \mu_1\sigma_2 dt\, dB_{2t}^H + \mu_2\sigma_1 dt\, dB_{1t}^H)$ so:

$$dS_{1t}S_{2t} = 2\rho\sigma_1\sigma_2 S_{1t}S_{2t}Ht^{2H-1}dt + c(dt). \tag{6}$$

Substitute the equation (5) until equation (6) to equation (4) with $dt \to 0$ then obtained:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}2H\sigma_1^2 S_1^2 t^{2H-1}dt + \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}2H\sigma_2^2 S_2^2 t^{2H-1}dt + \frac{\partial^2 V}{\partial S_1\, \partial S_2}2\rho\sigma_1\sigma_2 S_{1t}S_{2t}Ht^{2H-1}dt \tag{7}$$

given assumption Π is a risky portofolio. With the consideration of replacing the portofolio Π with one unit option V and unit $-\Delta$ of the underlying stock, then the value of the portofolio change at time t is given as $\Pi_t = V_t - \sum_{i=1}^{2}\Delta_i S_{it}$, when the portofolio is affected by changes in time (dt) then the value of portofolio changes into:

$$d\Pi_t = dV_t - d\Delta_1 S_{1t} - d\Delta_2 S_{2t} \tag{8}$$

So that the risk-free portofolio is made by implementing the arbitrage-free opportunity principle, then adjust,then adjust Δ and make Π a risk-free variable at a time interval $(t, t + dt)$. Assume that Δ does not change over time at that interval, so that:

$$d\Pi_t = r\Pi_t dt \tag{9}$$

Substitute equation (8) into equation (9), then obtained:

$$d(V_t - \Delta_1 S_{1t} - \Delta_2 S_{2t}) = r(V_t - \Delta_1 S_{1t} - \Delta_2 S_{2t})dt \tag{10}$$

Substitute equation (7) into equation (10), then obtained:

$$\begin{aligned}\left(\frac{\partial V}{\partial t} + rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} - \Delta_1 rS_1 - \Delta_2 rS_2\right)dt + \left(\sigma_1\frac{\partial V}{\partial S_1}dB_{1t}^H - \Delta_1\sigma_1 dB_{1t}^H\right) \\ + \left(\sigma_2\frac{\partial V}{\partial S_2}dB_{2t}^H - \Delta_2\sigma_2 dB_{2t}^H\right) + \frac{\partial^2 V}{\partial S_1^2}H\sigma_1^2 S_1^2 t^{2H-1}dt \\ + \frac{\partial^2 V}{\partial S_2^2}H\sigma_2^2 S_2^2 t^{2H-1}dt + 2\frac{\partial^2 V}{\partial S_1\, \partial S_2}\rho\sigma_1\sigma_2 S_{1t}S_{2t}Ht^{2H-1}dt \\ = r(V_t - \Delta_1 S_1 - \Delta_2 S_2)dt\end{aligned} \tag{11}$$

In order to portofolio Π to be risk free at $(t, t + dt)$, the coefficient of dB_t^H $(i = 1,2)$ must be zero.Furthermore, for example $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$ then equation (11) can be changed to:

$$\begin{aligned}\left(\frac{\partial V}{\partial t} + H\frac{\partial^2 V}{\partial S_1^2}\sigma_1^2 S_1^2 t^{2H-1}dt + H\frac{\partial^2 V}{\partial S_2^2}\sigma_2^2 S_2^2 t^{2H-1} + 2\frac{\partial^2 V}{\partial S_1\, \partial S_2}\rho\sigma_1\sigma_2 S_{1t}S_{2t}Ht^{2H-1}\right)dt \\ = r(V_t - \Delta_1 S_1 - \Delta_2 S_2)dt\end{aligned}$$

The problem of the Asian Rainbow put option price searching from V function, which satisfies to the following boundary conditions:

$$V(S_1, S_2, I_1, I_2, T) = \max\{K - min[G_1(T), G_2(T)], 0\} \tag{12}$$

The equation (12) is an Asian Rainbow option price which dependson the minimum geometric average value of the two underlying assets.

III. PRICING FORMULA OF GEOMETRIC ASIAN RAINBOW PUT OPTIONS

Let $V(\cdot)$ be the geometric Asian rainbow option price at time t which depends on S_1 and S_2. Given the boundary condition of the options on equation (12). Next, apply the following transformations:

$$y_i = \frac{\ln S_i}{T} + \frac{T-t}{T}\ln S_i, \qquad (i = 1,2) \tag{13}$$

$$z_i = y_i + \frac{r(T-t)^2}{2T} + \frac{H\sigma_i^2(T^{2H+1} - t^{2H+1})}{T(2H+1)} - \frac{T^{2H} - t^{2H}}{2}\sigma_i^2, \qquad (i = 1,2) \tag{14}$$

$$x_i = e^{z_1 - \left(r - \frac{1}{2}\sigma_i^2\right)\tau}, \qquad (i = 1,2) \tag{15}$$

$$\tau = \frac{2H}{T^2}\left(\frac{T^{2H} - t^{2H}}{2H}T^2 - 2T\frac{T^{2H+1} - t^{2H+1}}{2H+1} + \frac{T^{2H+2} - t^{2H++2}}{2H+2}\right) \tag{16}$$

The option price can be written as:

$$V(S_1, S_2, T) = e^{-r(T-t)}e^{r\tau}.F(x_1, x_2, \tau) \tag{17}$$

The Black-Scholes Model for the multi-asset option price is stated as follows:

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma_1^2 x_1^2\frac{\partial^2 F}{\partial x_1^2} + \rho\sigma_1\sigma_2 x_1 x_2\frac{\sigma^2 F}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 x_2^2\frac{\partial^2 F}{\partial x_2^2} + r x_1\frac{\partial F}{\partial x_1} + \frac{r x_2 \partial F}{\partial x_2} - rF \tag{18}$$

Equations (12) can also be transformed into:

$$F(z_1, z_2, 0) = \max\{K - min[x_1, x_2], 0\}. \tag{19}$$

The solution of the two-dimensional diffusion equation in equations (18) and (19) is:

$$\begin{aligned} F(x_2, x_2, \tau) = Ke^{-r\tau}N&\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r - \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{K}\right) + \left(r - \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}\right]; \rho\right) \\ - x_1 N&\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r + \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{x_1}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_2 - \sigma_1}{\sigma_{12}}\right) \\ - x_2 N&\left(-\left[\frac{\ln\left(\frac{x_2}{K}\right) + \left(r + \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln\left(\frac{x_1}{x_2}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_1 - \sigma_2}{\sigma_{12}}\right). \end{aligned}$$

where $N(\alpha, \beta; \theta)$ is the bivariate cumulative standard normal distribution with upper limits of integration α and β, and the correlation coefficient of θ. According to equation (17) by considering the two-dimensional diffusion equation above it becomes:

$$\begin{aligned} V(S_1, S_2, I_1, I_2, T) \\ = e^{-r(T-t)}KN&\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r - \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{K}\right) + \left(r - \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}\right]; \rho\right) \\ - e^{r\tau - r(T-t)}x_1 N&\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r + \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{x_1}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_2 - \sigma_1}{\sigma_{12}}\right) \\ - e^{r\tau - r(T-t)}x_2 N&\left(-\left[\frac{\ln\left(\frac{x_2}{K}\right) + \left(r + \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln\left(\frac{x_1}{x_2}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_1 - \sigma_2}{\sigma_{12}}\right) \end{aligned} \tag{20}$$

by supposing some equations as follows:

1) $$\begin{aligned} e^{r\tau - r(T-t)}x_i &= e^{r\tau - r(T-t)}e^{z_1 - \left(r - \frac{1}{2}\sigma_i^2\right)\tau}, \\ &= e^{r\tau - r(T-t)}e^{y_i + \frac{r(T-t)^2}{2T} + \frac{H\sigma_i^2\left(T^{2H+1} - t^{2H+1}\right)}{T(2H+1)} - \frac{T^{2H} - t^{2H}}{2}\sigma_i^2 - \left(r - \frac{1}{2}\sigma_i^2\right)\tau}, \\ &= e^{r\tau - r(T-t)}e^{\frac{I_i}{T} + \frac{T-t}{T}\ln S_i + (T-t)\left[\frac{r(T-t)}{2T} + \frac{H\sigma_i^2\left(T^{2H+1} - t^{2H+1}\right)}{T(2H+1)(T-t)} - \frac{T^{2H} - t^{2H}}{2(T-t)}\sigma_i^2\right] - \left(r - \frac{1}{2}\sigma_i^2\right)\tau} \end{aligned} \tag{21}$$

2) $$r_i^* = \frac{r(T-t)}{2T} + \frac{H\sigma_i^2\left(T^{2H+1} - t^{2H+1}\right)}{T(2H+1)(T-t)} - \frac{T^{2H} - t^{2H}}{2(T-t)}\sigma_i^2$$

Substitute r_i^* into the equation (21),

$$e^{r\tau - r(T-t)}x_i = e^{r\tau - r(T-t)}e^{\frac{I_i}{T} + \frac{T-t}{T}\ln S_i + r_i^*(T-t) - \left(r - \frac{1}{2}\sigma_i^2\right)\tau} \tag{22}$$

3) $S_i^* = e^{\frac{I_i}{T} + \frac{T-t}{T}\ln S_i + r_i^*(T-t)}$

Substitute S_i^* into the equation (22),

$$e^{r\tau - r(T-t)}x_i = e^{r\tau - r(T-t)}S_i^* e^{-\left(r - \frac{1}{2}\sigma_i^2\right)\tau},$$

$$= S_i^* e^{\frac{1}{2}\sigma_i^2\tau - r(T-t)}. \tag{23}$$

4) $\sigma_i^* = \sigma_i\sqrt{1 - \frac{4H(T^{2H+1} - t^{2H+1})}{T(2H+1)(T^{2H} - t^{2H})} + \frac{H(T^{2H+2} - t^{2H+2})}{T^2(H+1)(T^{2H} - t^{2H})}}, \quad (i = 1,2,12)$

$$\frac{\ln\left(\frac{x_i}{K}\right) + \left(r + \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}} = \frac{\ln(x_i) - \ln(K) + \left(r + \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}},$$

$$= \frac{\ln(S_i^*) - \left(r - \frac{1}{2}\sigma_i^2\right)\tau - \ln(K) + \left(r + \frac{1}{2}\sigma_i^2\right)\tau}{\sigma_i\sqrt{\tau}},$$

$$= \frac{\ln(S_i^*) - \ln(K) + \sigma_i^2\tau}{\sigma_i\sqrt{\tau}},$$

$$= \frac{\ln\left(\frac{S_i^*}{K}\right) + \sigma_i^2\tau}{\sigma_i\sqrt{\tau}}. \tag{24}$$

$$\frac{\ln(x_2/x_1) - \frac{1}{2}\sigma_{12}^2\tau}{\sigma_{12}\sqrt{\tau}} = \frac{\ln\left(\frac{S_2^*}{S_1^*}\right) + \frac{1}{2}(-\sigma_1^2 - \sigma_2^2 + 2\rho\sigma_1\sigma_2 - \sigma_1^2 + \sigma_2^2)\tau}{\sigma_{12}\sqrt{\tau}},$$

$$= \frac{\ln\left(\frac{S_2^*}{S_1^*}\right) + \frac{1}{2}(2\rho\sigma_1\sigma_2 - 2\sigma_1^2)\tau}{\sigma_{12}\sqrt{\tau}},$$

$$= \frac{\ln\left(\frac{S_2^*}{S_1^*}\right) + (\rho\sigma_1^*\sigma_2^* - \sigma_1^{*2})(T^{2H} - t^{2H})}{\sigma_{12}^*\sqrt{(T^{2H} - t^{2H})}}. \tag{25}$$

Substitute equation (21) until equation (25) into equation (20), so that the Asian Rainbow put option price formula using geometric averages for two assetsis obtained.

$$V(S_1, S_2, I_1, I_2, T) = e^{-r(T-t)}KN\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r - \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{K}\right) + \left(r - \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}\right]; \rho\right)$$
$$- e^{r\tau - r(T-t)}x_1 N\left(-\left[\frac{\ln\left(\frac{x_1}{K}\right) + \left(r + \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\left(\frac{x_2}{x_1}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_2 - \sigma_1}{\sigma_{12}}\right)$$
$$- e^{r\tau - r(T-t)}x_2 N\left(-\left[\frac{\ln\left(\frac{x_2}{K}\right) + \left(r + \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln\left(\frac{x_1}{x_2}\right) - \frac{\sigma_{12}^2\tau}{2}}{\sigma_{12}\sqrt{\tau}}\right]; \frac{\rho\sigma_1 - \sigma_2}{\sigma_{12}}\right),$$

IV. MONTE CARLO SIMULATION FOR ASIAN RAINBOW PUT OPTION PRICINGUSING GEOMETRIC AND ARITHMETIC AVERAGES

The Monte Carlo simulation method was chosen to present the simulation results which were then compared to the analytical results (manual calculations). In the above explanation, the Asian Rainbow put option price formulafor two assets using geometric averages has been analyzed analytically. For the Asian Rainbow put option price using arithmetic average do not have an analytical solution,so the simulation results here will be presented briefly for the Asian rainbow option price using geometric and arithmetic averages in graphics form. Set the initial parameters by supposing the initial asset price for$S_1$and$S_2$is\$40with an option maturity of four months, where$T - t = 1/3$and $H \in [0,1]$, then for the other parameter prices will be shown in Table 1. Suppose the initial time is $0 = t_0 < t_1 < \cdots < t_n = T$, and time intervals $\Delta t = \frac{T}{n}$. Based on the measurement of the risk neutral probability, a normal bivariate distribution for the stock price coefficientμ_i(i = *1,2*can be replaced by a risk free interest rate$r$. For the mean*r−12σi2Δt*, the variance*σi2Δt*, and the correlation coefficientρ.The simulation process is carried out through the following steps:

1. Put the generator component into the stock price model under the influence of fBm.

$$S_1(t_j) = S_1(t_{j-1}) \exp\left[\left(r(\Delta t)^{2H} - \frac{1}{2}\sigma_1^2(\Delta t)^{2H} + \sigma_1(\Delta t)^H\right)x_j\right],$$

$$S_2(t_j) = S_2(t_{j-1}) \exp\left[\left(r(\Delta t)^{2H} - \frac{1}{2}\sigma_2^2(\Delta t)^{2H} + \sigma_2(\Delta t)^H\right)y_j\right].$$

2. Calculate the average of the underlying assets that have been obtained from step (11) using this formula:

$$G_i(T) = \prod_{j=1}^{n} S_i\left(\frac{jT}{n}\right)^{\frac{1}{n}} \quad (i = 1,2),$$

for geometric averages. Meanwhile, for arithmetic averages use the following formula:

$$A_i(T) = \frac{1}{n}\sum_{i=0}^{n} S_i \ (i = 1,2).$$

for arithmetic averages, only numerical solutions are available because they do not satisfiy the Ito process requirements.

3. Run a simulation of the Asian rainbow put option prices with geometric and arithmetic averages using the following formula:

$$P_k(T) = e^{-rT}\max\left(K - min\left(G_1(T), G_2(T)\right), 0\right),$$
$$P_k(T) = e^{-rT}\max\left(K - min\left(A_1(T), A_2(T)\right), 0\right).$$

4. Repeat step (1) until step (3) Ntimes to get more accurate results, then the simulation results for$N$times are averaged using the formula:

$$P(T) = \frac{1}{N}\sum_{k=1}^{N} P_k(T).$$

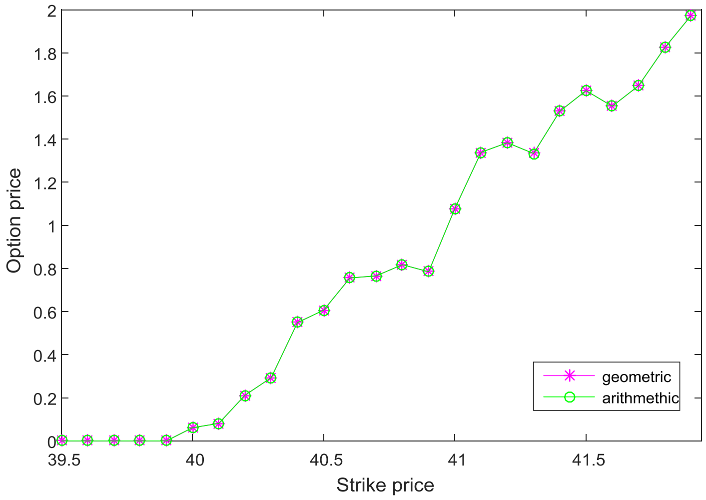

Figure1 The comparison of Asian Rainbow put option pricing using geometric and arithmetic averages

Based on Figure 1 above and Table 1 belowusing$N = 100$and$n = 88$, itget the result that for analytical and simulation results for Asian Rainbow put option pricing using geometric averages it get similar results. Meanwhilefor the results comparison of Asian Rainbow put option pricing using geometric and arithmetic averages obtainedalmost the same results or can be said the results have a small error.Figure 1 also shows that the Asian rainbow put option pricehas increased with the increasing in strike price.According to the put option principle that the greater the strike price, the greater the option price. Or it could be said the increase in the strike price is directly proportional to the increase in the option price

Table1 The comparison of analytical results and Monte Carlo simulation for Asian Rainbow put option pricing using geometric and arithmetic averages.

No	σ_1	σ_2	K	r=0.03			r=0.05			r=0.07				
				Analytical value (G)	Simulated value (G)	Selisih	Analytical value (G)	Simulated value (G)	Selisih	Analytical value (G)	Simulated value (G7)	Selisih	Simulated Value (A7)	(A7)-(G7)
	ρ=-0.3													
1	0,2	0,3	35	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0000	0,0000
2	0,2	0,3	40	0,0090	0,0503	0,0413	0,0090	0,0409	0,0319	0,0090	0,0606	0,0516	0,0527	0,0079
3	0,2	0,3	45	4,7478	4,9950	0,2472	4,6150	4,9483	0,3333	4,4817	4,9373	0,4556	4,9297	0,0076
4	0,2	0,4	35	0,0089	0,0000	0,0089	0,0089	0,0000	0,0089	0,0089	0,0000	0,0089	0,0000	0,0000
5	0,2	0,4	40	0,0089	0,0545	0,0456	0,0089	0,0695	0,0606	0,0089	0,0582	0,0493	0,0534	0,0048
6	0,2	0,4	45	4,7138	4,9945	0,2807	4,5809	4,9791	0,3982	4,4475	4,9375	0,4900	4,9283	0,0092
7	0,3	0,4	35	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0000	0,0000
8	0,3	0,4	40	0,0090	0,0819	0,0729	0,0090	0,0710	0,0620	0,0090	0,0742	0,0652	0,0728	0,0014
9	0,3	0,4	45	4,6898	5,0130	0,3232	4,5567	4,9796	0,4229	4,4232	4,9510	0,5278	4,9315	0,0195
	ρ =0.1													
10	0,2	0,3	35	0,0089	0,0000	0,0089	0,0089	0,0000	0,0089	0,0089	0,0000	0,0089	0,0000	0,0000
11	0,2	0,3	40	0,0089	0,0442	0,0353	0,0089	0,0398	0,0309	0,0089	0,0571	0,0482	0,0435	0,0136
12	0,2	0,3	45	4,7478	4,9790	0,2312	4,6149	4,9418	0,3269	4,4816	4,9290	0,4474	4,9169	0,0121
13	0,2	0,4	35	0,0087	0,0000	0,0087	0,0087	0,0000	0,0087	0,0088	0,0000	0,0088	0,0000	0,0000
14	0,2	0,4	40	0,0087	0,0638	0,0551	0,0087	0,0577	0,0490	0,0088	0,0664	0,0576	0,0658	0,0006
15	0,2	0,4	45	4,7137	5,0009	0,2872	4,5807	4,9640	0,3833	4,4473	4,9412	0,4939	4,9439	0,0027
16	0,3	0,4	35	0,0089	0,0000	0,0089	0,0089	0,0000	0,0089	0,0090	0,0000	0,0090	0,0000	0,0000
17	0,3	0,4	40	0,0089	0,0890	0,0801	0,0089	0,0656	0,0567	0,0090	0,0704	0,0614	0,0737	0,0033
18	0,3	0,4	45	4,6897	5,0264	0,3367	4,5566	4,9667	0,4101	4,4231	4,9439	0,5208	4,9440	0,0001
	ρ =0.5													
19	0,2	0,3	35	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0000	0,0000
20	0,2	0,3	40	0,0090	0,0510	0,0420	0,0090	0,0368	0,0278	0,0090	0,0468	0,0378	0,0538	0,0070
21	0,2	0,3	45	4,7479	4,9884	0,2405	0,0090	4,9352	4,9262	0,0090	4,9187	4,9097	4,9267	0,0080
22	0,2	0,4	35	0,0000	0,0000	0,0000	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0000	0,0000
23	0,2	0,4	40	0,0000	0,0579	0,0579	0,0090	0,0460	0,0370	0,0090	0,0641	0,0551	0,0678	0,0037
24	0,2	0,4	45	4,7139	4,9925	0,2786	4,6150	4,9413	0,3263	4,4817	4,9296	0,4479	4,9411	0,0115
25	0,3	0,4	35	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0090	0,0000	0,0090	0,0000	0,0000
26	0,3	0,4	40	0,0090	0,0630	0,0540	0,0090	0,0667	0,0577	0,0090	0,0620	0,0530	0,0699	0,0079
27	0,3	0,4	45	4,6898	4,9878	0,2980	4,6150	4,9665	0,3515	4,4817	4,9306	0,4489	4,9412	0,0106

V. SENSITIVITY ANALYSIS

As an evaluation from the results of the formula presented, the next step is to conduct a sensitivity analysis of each parameter, such as correlation coefficient parameters, Hurst index, and risk-free interest rates. This step aims to determine whether the parameters used have a significant effect in influencing changes in option prices.

A. Sensitivity Analysisfor Correlation Coefficient

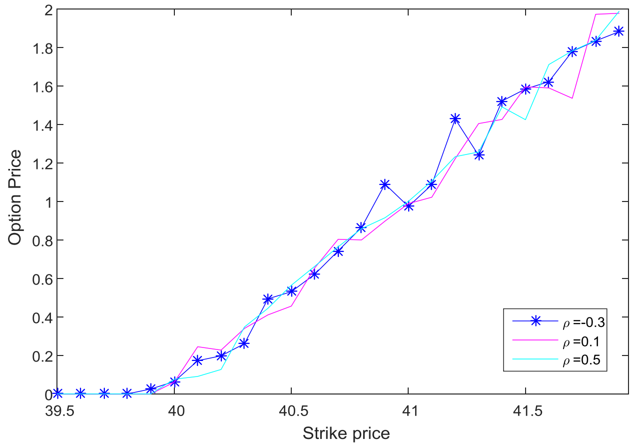

Figure 2 Asian Rainbow put option price using differentρvalue

Figure 2 shows the stock price change at the strike price$K = \$39.5, \dots, \$42$, where the initial stock price of the two underlying assets is\$40, volatility$\sigma_1 = 0.2, \sigma_2 = 0.3$, Hurst index 0.7, and interest rates$r = 0.03$, with different correlation coefficients of$-0.3, 0.1, 0.5$. The greater the correlation coefficient of assetsρ, the greater the probability that the two underlying assets will rise or fall simultaneously. Therefore the Asian Rainbow put option is based on which option price is greater. From Figure2 it is found that the correlation coefficient change does not significantly affect to the Asian Rainbow put options price.

B. Sensitivity Analysisfor Hurst Index

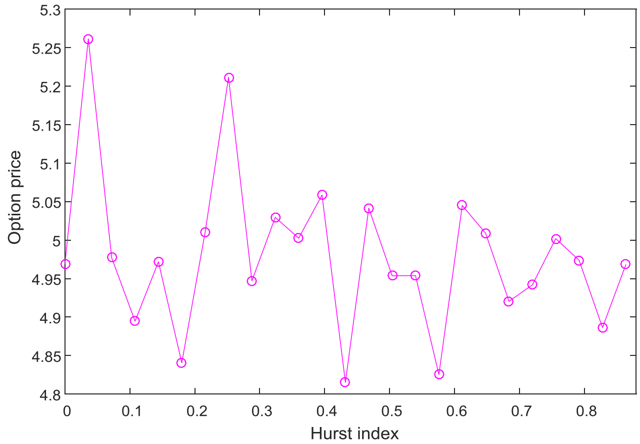

Figure 3 Asian Rainbow put option prices for changes in different Hurst index values

Figure3shows the change in the price of the Asian rainbow put option against the change in the value of the Hurst index. For the case of the Hurst index value$H > 0.5$where in the fractional Brownian motion process captures a case of self-similarity which can also be seen in Figure 3 that there are several option prices that tend to be the same (the difference is not too far) compared to option prices when the Hurst index value is$0 \leq H \leq 0.5$. Therefore fractional Brownian motion is suitable to be applied in modeling the Asian rainbow put stock option price, because it can catch or sensitive to changes that are fluctuating and tend to be self-similarity.

C. Sensitivity Analysis untuk for Risk-Free Interest Rate

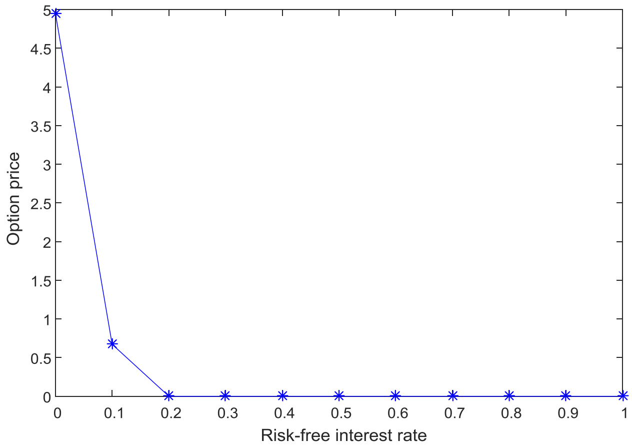

Figure4 Asian Rainbow put option price using an increase in $r$value of 10%

Figure 4 shows the results of the sensitivity analysis for the Asian Rainbow put option price to the risk-free interest rate parameter with a 10% increase. Figure 4also shows that the Asian Rainbow put option price has decreased in value with increasing risk-free interest rates. It can be seen when $t_0$to$t_2$when the option values decreases until whent_2 to t_{10}which is 0 (zero) or can be called convergent. This phenomenon shows that the increase in interest rates is inversely proportional to the value of the Asian rainbow put option price which causes the value of the option price is decreasing.

VI. CONCLUSSION

The comparison between analytical and simulation results using Monte Carlo for Asian Rainbow put option prices based on fractional Brownian motion using geometric averages obtained almost the same results while for comparison of results between Asian Rainbow put option prices using geometric and arithmetic averages results obtained relativesame.In this study, fractional Brownian motion has not been analyzed using arithmetic averages analytically. The discussion is only based on numerical simulations. For this reason, in subsequent studies, it can be further studied in analytical completion. The research can also be continued for other option models, such as European options and American options.

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