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# Fractional Derivatives and Theorem of Hardy and Littlewood

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## I. INTRODUCTION

This paper gives a unified account of a body of work on Hardy-Littlewood theorem of functions regular in the unit disc, relating Particularly to the fractional derivatives and integrals of such functions. Twotypesoffractionalderivativeand integral arediscussed.Foreachofthetwotypesoffractionalderivativeconsidered,a function analogous to the Littlewood-Paley $g$-function is defined,and the properties of these two $g$-type functions are discussed. The results obtained here include Several new inequalities, and, inparticular,an extension (Theorem5)of a theorem of Hirschman for indices less than or equalto1.

The remaining contents are as follows.In§ 4the Hardy-LittlewoodMaximal theorem is applied to obtain an inequality for fractional derivatives.In§10 an auxiliary theorem equivalent to one of Hardy and Littlewood is proved, and this is used to obtain anew proof of a theorem on majorants.In§§11-12 new proofs of the Hardy-Little wood theorem on fractional integrals and of some related results are given,and in§13a theorem ofHardyandLittlewoodontheconvolutionseriesoftwo Powerseriesiscompletedandextended.Theresultsobtainedhaveobviousapplicationsintheclassical theoryofFourierseries,viaM.Riesz'stheorem on conjugatefunctions, butthesearenotstatedexplicitly.

**2.**We assume throughout this paper that $\varphi$ is a function regular in the unit disc$\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and that$\varphi(z) = \sum_{\epsilon=0}^{\infty} c_{1+\epsilon} z^{1+\epsilon}$ $(z \in \Delta)$.

We write

$$M_{1+\epsilon}(\varphi, 1-\epsilon)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\varphi(1-\epsilon)e^{i\theta}\right|^{1+\epsilon} d\theta\right\}^{\frac{1}{1+\epsilon}} (\epsilon \geq 0)$$

$$M(\varphi, 1-\epsilon) = M_{+\infty}(\varphi, 1-\epsilon) = \sup_{\theta}\left|\varphi\left((1-\epsilon)e^{i\theta}\right)\right|$$

It is familiar that if $(\epsilon \geq 0)$, then $M_{1+\epsilon}(\varphi, 1-\epsilon)$ increases with $(1-\epsilon)$, and therefore tends to a finite limit or $+\infty$ as $\epsilon \to 2$. We define

$$\mu(1+\epsilon)(\varphi) = \lim_{\epsilon \to 2} M_{1+\epsilon}(\varphi, 1-\epsilon)(\epsilon \geq 0) \qquad (2.1)$$

the value$+\infty$ being allowed. The class of $\varphi$ for which the limit in (2.1) is finite is, of course, the class $H^{1+\epsilon}$. It is familiar that if $\varphi \in H^{1+\epsilon}$then$\varphi$has a radial limit$\varphi\left(e^{i\theta}\right) = \lim_{\epsilon \to 2} \left((1-\epsilon)e^{i\theta}\right)$ for almost all$\theta$, and that

$$\mu_{1+\epsilon}(\varphi) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left|\varphi\left(e^{i\theta}\right)\right|^{1+\epsilon} d\theta\right|\right\}^{\frac{1}{1+\epsilon}} (\epsilon \geq 0)$$

For any real or complex – valued function $f$measurable in the interval$[-\pi, \pi]$ we write

$$\mu_{1+\epsilon}(f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\varphi)|^{1+\epsilon} d\theta\right\}^{\frac{1}{1+\epsilon}} (\epsilon \geq 0)$$

$$N_{1+\epsilon}(\varphi) = \mu_{+\infty}(f) = ess \sup_{\theta}|f|$$

the value$+\infty$ being allowed. The class of$f$ for which$N_{1+\epsilon}(f)$ is finite (where$\epsilon \geq 0$) is the class$L^{1+\epsilon}(-\pi, \pi)$. For any number$1+\epsilon$used as an index (exponent) and such that $\epsilon \geq 0$, we write$\epsilon = 0$, so that $1+\epsilon$and $\frac{1+\epsilon}{\epsilon}$ are conjugate indices in the sense of Hölder's inequality we extend this notation to include $\epsilon = 0$and $\epsilon = +\infty$.

Any inequality $L \leq R$ quoted or proved is to be interpreted as meaning if $R$ is finite, the $L$ is finite, and $L \leq R$. We use $A(b, c, \ldots)$ to denote a positive constant depending only on $b, c, \ldots$, not necessarily the same on any two occurrences, $A$ by itself will denote a positive absolute constant. We also sometimes write $B$ for constant of the form $A(b, c, \ldots)$, these too are not necessarily the same on any two occurrences. We need the known theorems.
**Theorem A.** Let $0 \leq \varepsilon \leq \infty$, Let $f, g$ be real-or complex – valued functions measurable on $[-\pi, \pi]$, and let

$$h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t) g(t)\ dt$$

Then

$$N_{\frac{1+\epsilon}{1-\epsilon}}(h) \leq N_{1+\epsilon}(f) N_{1+\epsilon}(g)$$

This is a well – known inequality of W-H Young.(see for example,[22,i.p.37].
**Theorem B.** Let $f$ be a function measurable on the interval $(0, +\infty)$, let $f(x) \geq 0$ for $x > 0$ and let $F_\delta(x)$ be the Riemann – Liouville integral of $f$ of order $\delta$ with origin $o$, *i.e.*

$$F_\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x - y)^{\delta - 1} f(y)\ dy$$

If $\epsilon > 0$ and either

$$\epsilon \geq 0, \delta > \frac{1 + 2\epsilon}{\epsilon}, \text{or } \epsilon > 0, \delta = \frac{1 + \epsilon}{\epsilon}$$

Then

$$\left\{ \int_0^{+\infty} x^{-(1+\epsilon(1+2\epsilon)\delta)} F_\delta^{1+2\epsilon}(x) dx \right\} \leq A(1 + 2\epsilon; 1 + \epsilon, \delta, \epsilon - 1) \left\{ \int_0^{+\infty} x^{\epsilon^2} f^{(1+\epsilon)}(x) dx \right\}^{\frac{1}{1+\epsilon}}$$

For $\delta > \frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}$ this is essentially an elementary application of Hölder's inequality, for $\delta > \frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}$, then result lies deeper, the case $\epsilon = 0$ being the Hardy-Littlewood theorem on functional integrals of real function (see [5,Th.2]).
**Theorem C.** If $\varphi \in H^{1+\epsilon}$, where $\epsilon \geq 0$, then $\varphi$ can be expressed in the form $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ and $\varphi_2$ are regular and have no zeros in $\Delta$, and

$$\mu_{1+\epsilon}(\varphi_i) \leq 2N_{1+\epsilon}(\varphi) \quad (i = 1,2)$$

This is a familiar theorem of Hardy and Littlewood ( [8,p.207]).
**Theorem D.** If $\epsilon \geq 0$ and $\mu = \max[0, \frac{1}{\epsilon}]$, then

$$(1 + \epsilon)^{-\mu} |C_{1+\epsilon}| \leq A(1 + \epsilon) N_{1+\epsilon}(\varphi)$$

This also is due to Hardy and Littlewood ([12,Theorem28]).
**Theorem E.** Let $0 < \epsilon < 1$, and let $S(\theta) = S_{1-\epsilon}(\theta)$ be the open subset of $\Delta$ bounded by the two tangents from the point $e^{i\theta}$ to the circle with center $o$ and radius $1 - \epsilon$, together with the longer arc of this circle between the points of contact. Let also $\varphi$ be regular in $\Delta$ and let

$$\Phi(\theta) = \sup_{z \in S(\theta)} |\varphi(z)|$$

then for $\epsilon \geq 0$ $\mu_{1+\epsilon}(\Phi) \leq A(1 - \epsilon, 1 - \epsilon) \mu_{1+\epsilon}(\varphi)$ This is the Hardy – Littlewood complex Max′ theorem (see, forexample,[22,i,p.278]).
**Theorem F.** Let $\varphi$ be regular in $\Delta$ and let

$$T_{1+2\epsilon,\sigma}(\theta) = \left\{ \int_0^1 (\epsilon)^{\sigma + 2\epsilon - 1}\ d1 - \epsilon \int_{-\pi}^{\pi} \frac{\left|\varphi'(1 - \epsilon) e^{i\theta - it} dt\right|^{1+2\epsilon}}{|1 - \epsilon e^{it}|^\sigma} \right\}^{\frac{1}{1+2\epsilon}}$$

If $\epsilon > -1, \sigma > \max\left\{1, \frac{1+2\epsilon}{1+\epsilon}\right\}$ then

$$\mu_{1+\epsilon}(T_{1+2\epsilon,\sigma}) \leq A(1 + 2\epsilon, 1 + \epsilon, \sigma) \mu_{1+\epsilon}(\varphi).$$

This is one of the consequences of the Littlewood – Paley g- theorem. (see[3,Th.15]).
**Theorem G.** Let $f \in L^{1+\epsilon}(-\pi, \pi)$, where $\epsilon > 0$, let the complex Fourier series of $f$ be $\sum_{-\infty}^{\infty} \gamma\ e^{ni\theta}$, and let

$$\psi(z) = \sum_{\epsilon=0}^{\infty} \gamma_{1+\epsilon}\ z^{1+\epsilon} (z \in \Delta).$$

Then

$$\mu_{1+\epsilon}(\psi) \leq A(1 + \epsilon) N_{1+\epsilon}(f)$$

This is equivalent to $M$. Riesz's theorem on conjugate functions. (see Hardy and Littlewood[9] for futher explations).

In addition to these theorems we also make extensive use of Hölder's inequality, and of Minkowski's inequality in the form

$$\left\{g(x)dx\left\{\int f(x,y)h(y)dy\right\}^{1+2\epsilon}\right\}^{\frac{1}{1+2\epsilon}} \leq \int h(y)dy\left\{\int f^{1+2\epsilon}(x,y).g(x)\,dx\right\}^{\frac{1}{1+2\epsilon}}$$

Where$\epsilon \geq 0$ and $f, g, h$are nonnegative. We use also the analogous result for $\epsilon = +\infty$, namely

$$\sup_x\left\{\int f(x,y)h(y)\ dy\right\} \leq \int\left\{\sup_x f(x,y)\right\} h(y)\,dy$$

**3.Fractional derivatives and integrals:**The definition $f$ fractionalderivativeandintegralwhichisusedin§§3-13 is asfollows.

Let$\varphi$ is regular in $\Delta$, and

$$\varphi(z) = \sum_{\epsilon=-1}^{\infty} c_{1+\epsilon} z^{1+\epsilon} \quad (z \in \Delta)$$

Then for any $\epsilon \geq -1$ the fractional derivative $v^{1+\epsilon}\varphi$ of $\varphi$ of order$1+\epsilon$ is given by

$$v^{1+\epsilon}\varphi(z) = \sum_{\epsilon=-1}^{\infty}(1+\epsilon)^{1+\epsilon}c_{1+\epsilon}z^{1+\epsilon}(z \in \Delta) \qquad (3.1)$$

Clearly$v^{1+\epsilon}\varphi$ is regular in $\Delta$, and

$$v^{1+\epsilon}(v^{\gamma}\varphi) = v^{1+\epsilon+\gamma}\varphi \qquad (3.2)$$

For all nonnegative $1+\epsilon, \gamma$

The corresponding definition of the fractional integral applies only to functions vanishing at the origin. Thus if$\varphi(0) = c_0 = 0$, then for any $\epsilon \geq -1$the fractional integral $v_{1+\epsilon}\varphi$ of $\varphi$ of order $1+\epsilon$is given by

$$v_{1+\epsilon}\varphi(z) = \sum_{\epsilon=0}^{\infty}(1+\epsilon)^{-(1+\epsilon)}c_{1+\epsilon}z^{1+\epsilon}(z \in \Delta) \qquad (3.3)$$

As for fractional derivative, the fractional integral fractional $v_{1+\epsilon}\varphi$ is regular in $\Delta$ and

$$v_{1+\epsilon}(v_{\gamma}\varphi) = v_{1+\epsilon+\gamma}\varphi \qquad (3.4)$$

For all nonnegative $1+\epsilon, \gamma$

When $\varphi(0) = 0$ (3.1) and (3.3) can be used to define $v_{1+\epsilon}\varphi$ and $v^{1+\epsilon}\varphi$for all real$1+\epsilon$(so that $v_{1+\epsilon}\varphi = v^{-(1+\epsilon)}\varphi$ for all real $1+\epsilon$) and then (3.2) and (3.4) hold for all real $1+\epsilon, \gamma$.

The functions $v_{1+\epsilon}\varphi$ and $v^{1+\epsilon}\varphi$ defined above seem to have been first studied by Hadamord [7] . For $\epsilon > -1, i^{-(1+\epsilon)}v_{1+\epsilon}\varphi\left((1+\epsilon)e^{i\theta}\right)$ is the Welyl fractional integral of order $\epsilon$of the function $\theta \to \varphi\left((1+\epsilon ei\theta\right.$, and for any positive integer $m$

$$i^m v^m\left((1+\epsilon)e^{i\theta}\right) = \frac{\partial m}{\partial\theta^m}\varphi\left((1+\epsilon)e^{i\theta}\right)$$

Thus the definitions (3.3) and (3.4) correspond roughly to differentiation and integration with respect to $\theta$. We note also that if $m$is apositive integer then

$$v^m\varphi(z) = \left(z\frac{d}{dz}\right)^m \varphi(z) \qquad (3.5)$$

So that $v^1$has its traditional meaning of $z\frac{d}{dz}$.

For $\epsilon > -1$the fractional integer $v_{1+\epsilon}\varphi$is connected with $\varphi$by the relation

$$v_{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right) = \frac{1}{\Gamma(1+\epsilon)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\epsilon}\left(\sigma\,e^{i\theta}\right)\frac{d\sigma}{\sigma} \quad (3.6)$$

Where $0 < \epsilon < 1$this relation is easily obtained by term integration, using the formulae

$$(1+\epsilon)^{-(1+\epsilon)}\int_0^{1+\epsilon}\left(\log\frac{1+\epsilon}{\sigma}\right)^{\epsilon}\sigma^{\epsilon}d\,\sigma\int_0^1\left(\log\frac{1}{\delta}\right)^{\epsilon}\delta^{\epsilon}d\delta = \int_0^{\infty}t^{\epsilon}e^{-(1+\epsilon)}dt$$
$$= (1+\epsilon)^{-(1+\epsilon)}\Gamma(1+\epsilon) \qquad (3.7)$$

where $\epsilon > -1$

The formula (3.6) was obtained by Hadamard [7,p.157],but does not seem to have been used by subsequent writers on fractional derivatives and integrals. In§§4-12 wedevelopthetheory ofthe functions $v^{1+\epsilon}\varphi$ and$v_{1+\epsilon}\varphi$, making systematic use of the formula (3.5).

**4.Application of Hadamard's formula (3.6), we prove :**

**Theorem 1.** Let$S_{1-\epsilon}(\theta)$ be the Kit – shaped region defined on Theorem E where$0 < \epsilon < 1$ let $\Phi(\theta) = \sup_{z\in S_{1-\epsilon}(\theta)}|\varphi(z)|$, and let $\epsilon > -1$. Then for $0 \leq \epsilon < 1$.

$$\left|v^{1+\epsilon}\varphi(1-\epsilon)e^{i\theta}\right| \leq A(1+\epsilon, 1-\epsilon)(1-\epsilon)(\epsilon)^{-(1+\epsilon)}\Phi(\theta) \qquad (4.1)$$

Asimilar result for a different type of fractional derivative isProved by Hardy and Littlewood [17,Th.5] (see also Hirschman [18,Lemma4.1],and Flett [6,Th.8]).Suppose first that $1+\epsilon$is positive integer,$m$ say and let $G$by the circle with centre $z=(1-\epsilon)e^{i\theta}$ and radius $\frac{1}{2}(1-\epsilon)(\epsilon)$. By (3.5), for $z\neq 0$we have

$$z^{-1}v^m\varphi(z)=z^{-1\left(z\frac{d}{dz}\right)^m}=\frac{1}{2\pi i}\int_c\frac{P(\zeta,z)\varphi(\zeta)}{(\zeta-z)^{m+1}}$$

where $P$is polynomial of degree $m-1$in $\zeta,z$depending only on $m$. Since$G\subset S_{1-\epsilon}(\theta)$, it follows that

$$(1+\epsilon)^{-1}\left|v^m\varphi\left((1-\epsilon)e^{i\theta}\right)\right|\leq A(1+\epsilon,1-\epsilon)(\epsilon)^{-m}\Phi(\theta)\qquad(4.2)$$

and this implies (4.1), since$v^m\varphi(0)=0$.

Nextlet$1+\epsilon$ be nonintegral, and let $m[1+\epsilon]+1$(where $[1+\epsilon]$denotes as usual, the integral part of $1+\epsilon$). Since$v^{1+\epsilon}\varphi=v_{m-(1+\epsilon)}(v^m\varphi)$. (4.1) gives

$$v^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)=\frac{1}{\Gamma\big(m-(1+\epsilon)\big)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{m-\epsilon}v^m\varphi\left(\sigma e^{i\theta}\right)\frac{d\sigma}{\sigma}\,(4.3)$$

and since $\log\frac{1}{x}\geq 1-x$for$x>0$, and $m-\epsilon\leq 2$, we obtain from (4.3)and (4.2)that

$$\left|v^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)\right|$$

$$\leq A(1+\epsilon,1-\epsilon)\Phi(\theta)(1-\epsilon)^{-m+\epsilon+2}\int_0^{1+\epsilon}(1-\epsilon-\sigma)^{m+\epsilon}(1-\sigma)^{-m}\,d\sigma\qquad(4.4)$$

On substituting $\sigma=1-\epsilon\,x$, we see that the integral -*on the right is equal to

$$(\epsilon)^{-(1+\epsilon)}\int_1^{\frac{1}{-\epsilon}}(x-2)^{m-\epsilon-2}x^{-m}\,dx\leq(\epsilon)^{-(1+\epsilon)}\int_1^{\infty}A(1+\epsilon)(\epsilon)^{-(1+\epsilon)}\qquad(4.5)$$

and (4.4)and (4.5)together imply (4.1)for $\frac{1}{2}\leq\rho<1$. On the other hand, if $0<\rho<\frac{1}{2}$, then the integral on the right of (4.4)does notexceed

$$2^m\int_0^{1+\epsilon}(1-\epsilon-\sigma)^{m-\epsilon-2}d\sigma=2^m(1-\epsilon)^{m-1-\epsilon}\big(m(1+\epsilon)\big)\leq A(1+\epsilon)(1-\epsilon)^{m-1-\epsilon}(\epsilon)^{-1-\epsilon}$$

and again the inequality (4.1) follows.

**THEOREM 1.COROLLARY 1.**If $-1<\epsilon\leq\infty$,, then for$0\leq\epsilon<1$,

$$M_{1+\epsilon}(v^{1+\epsilon}\varphi,1-\epsilon)\leq(1+\epsilon,1+\epsilon)(1-\epsilon)(\epsilon)^{-(1+\epsilon)}N_{1+\epsilon}(\varphi)$$

This follows from the main theorem and theorem E, with$\epsilon=\frac{1}{2},\eta=\frac{1}{2}$ (say). Applying this Corollary to the function $z\to\varphi\left((1-\epsilon)^{\frac{1}{2}}z\right)$, we deduce also

**THEOREM 1.COROLLARY 2.** Let $\varepsilon\geq 0$ and let

$$M_{1+\epsilon}(\varphi,1-\epsilon)\leq c(1-\epsilon)(0\leq\epsilon<1)$$

Then for$\epsilon>-1$

$$M_{1+\epsilon}(v^{1+\epsilon}\varphi,1-\epsilon)\leq A(1+\epsilon,1+\epsilon)(1-\epsilon)^{12}(\epsilon)^{-(1+\epsilon)}c\left((1-\epsilon)^{\frac{1}{2}}\right),(\epsilon\geq 0)$$

**5.Theorems of Littlewood –Paley type:** We consider next aGroup of three theorems closely related to results of Littlewood and Paley,Hirschman,and the [23].

For any $\varphi$ regular in the unit disc $\Delta$, and for any positive $1+\epsilon$, let

$$G_{1+2\epsilon,1+\epsilon}(\theta)=\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1}\left|v^{1+\epsilon}\left(\varphi(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}$$

**Theorem 2.** If $\epsilon>-1$, and either$\epsilon\geq 0,\delta>\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}$ or $\epsilon>0,\delta=\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}$ then for each $\theta$.

$$G_{1+2\epsilon,1+\epsilon}(\theta)\leq A(1+2\epsilon,1+\epsilon,1+\epsilon,\delta)G(1+\epsilon),1+\epsilon+\delta(\theta)\qquad(5.1)$$

In particular, if $\epsilon\geq 0$and $\gamma>1+\epsilon>0$, then for each $\theta$

$$G_{1+2\epsilon,1+\epsilon}(\theta)\leq A(1+2\epsilon,1+\epsilon,\gamma)G_{1+2\epsilon,\gamma}(\theta)\ (5.2)$$

**Theorem 3.** If $\epsilon>-1$, then

$$\mu_{1+\epsilon}\big(G_{1+2\epsilon,1+\epsilon}\big)\leq A(1+2\epsilon,1+\epsilon,\gamma)\mu_{1+\epsilon}(\varphi)\qquad(5.3)$$

**Theorem 4.**$0<\epsilon\leq\frac{1}{2}$ and $\varphi(0)=0$ then

$$\mu_{1+\epsilon}(\varphi)\leq A(1+2\epsilon,\epsilon+1,1+\epsilon)\mu_{1+\epsilon}\big(G_{1+2\epsilon,1+\epsilon}\big)$$

The results of Theorems 3and 4 with $G_{(1+2\epsilon)(1+\epsilon)}$ replaced by the function$g_{(1+2\epsilon)(1+\epsilon)}$given by

$$g_{(1+2\epsilon)(1+\epsilon)}=\left\{\int_0^1(\epsilon)^{(1+2\epsilon)(1+\epsilon)-1}(1-\epsilon)^{-(1+2\epsilon)}\left|v^{(1+\epsilon)}\varphi\left((1-\epsilon)e^{i\theta}\right)\right|^{(1+2\epsilon)}d(1-\epsilon)\right\}^{\frac{1}{1+2\epsilon}}\qquad(5.5)$$

Are already known. The cases $\epsilon = 0$, of these results for$g_{(1+2\epsilon)(1+\epsilon)}$were proved by Littlewoodand Paley [19],the function$g_{2,1}$being the well-known Littlewood- Paley $g$-function. The remaining cases where $\epsilon = 0$are due to Marcinkiewicz and Zygmund [20],and the cases where $\epsilon \neq 0$are due to Hirschman [18]and theAuthor[4,6].Thecrucialresultforthesetheoremsfor$g_{(1+2\epsilon)(1+\epsilon)}$ is that for $g_{2,1}$correspondingtoTheorem3(i.e.the Littlewoods-Paley(g-theorem),all the other results being obtainable from this.It is easy to pass from(5.3)to the corresponding inequality for $g_{(1+2\epsilon)(1+\epsilon)}$.And vice-versa,for it is obvious that if$\epsilon > 0$,

$$G_{(1+2\epsilon)(1+\epsilon)}(\theta) \leq A(1+2\epsilon, 1+\epsilon) g_{(1+2\epsilon)(1+\epsilon)}(\theta), \qquad (5.6)$$

and in virtue of Theorem 1,we have also

$$g^{(1+2\epsilon)}_{(1+2\epsilon)(1+\epsilon)}(\theta) = \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1} \leq A(1+2\epsilon, 1+\epsilon)\Phi^{1+2\epsilon}(\theta) + A(1+2\epsilon, 1+\epsilon) G^{(1+2\epsilon)}_{(1+2\epsilon)(1+\epsilon)}(\theta) \qquad (5.7)$$

for $\epsilon > -\frac{1}{2}$. ItisalsonotdifficulttodeduceTheorem4fromtheresultfor$g_{(1+2\epsilon)(1+\epsilon)}$corresponding to Theorem 3. However,the arguments involved in the proofs of these various results,at least for$\epsilon \neq 0$,apply much more naturally to$G_{(1+2\epsilon)(1+\epsilon)}$ than to$g_{(1+2\epsilon)(1+\epsilon)}$and it seems worth while to give independent proofs of Theorems 3 and4.The inequality (5.1)is new.It shows in particular that the cases $\epsilon \neq \frac{1}{2}, \epsilon \neq 0$ ofTheorems 3and 4are implied by the cases $\epsilon = \frac{1}{2}$ of these results, and thus provides a new proof of theresultsof Marcinkiewicz.Theimple specialcase(5.2)alsoenablesustoreducetheproofof Theorem3tothecasewhere$1+\epsilon$ isapositiveinteger,andthisinturn simplifiesone oftheestimatesinvolved.

**6. We prove Theorem 2:**If $\epsilon > -1, \delta > 0$ then $v^{1+\epsilon}\varphi = v_\delta(v^{1+\epsilon+\delta}\varphi)$, so that by (3.6)

$$\left|v^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)\right| \leq \frac{1}{\Gamma(\delta)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1}\left|v^{1+\epsilon+\delta}\varphi\left(e^{i\theta}\right)\right|\frac{d\sigma}{\sigma}$$

The required inequality (5.1) is therefore a consequence of the following Lemma

**Lemma 1.** Let $h$be a function measurable on the interval (0.1), let $h(1-\epsilon) \geq 0$ for$0 < \epsilon < 1$ and let

$$h_\delta(1-\epsilon) = \frac{1}{\Gamma(\delta)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1} h(\sigma)\frac{d\sigma}{\sigma}$$

If $\epsilon > -1$, andeither$\epsilon \geq 0, \delta > \frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}$or$\epsilon > 0, \delta = \frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}$

Then

$$\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} h_\delta^{1+2\epsilon}(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}$$

$$\leq A(1+2\epsilon, 1+\epsilon, (1+\epsilon), \delta)\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+\epsilon)(2+\epsilon)\delta-1} h^{1+\epsilon}(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}$$

This follows easily from Theorem B by transformation.

$$\frac{1}{x} = \log\frac{1}{1-\epsilon}, \frac{1}{y} = \log\frac{1}{\sigma}, f(x) = x^{-\delta-1}h\left(e^{-\frac{1}{x}}\right).$$

The lemma may also be proved independently of Theorem B. InOur arguments we make essential use only of the case $\epsilon = 0$ (thisGives the inequality (5.2)), and since the direct proof of this case ofThe lemma isparticularlysimple,wegiveithereforthesake ofcompleteness.

Let$\epsilon \geq 0, \delta > 0$, and choose $\mu$ depending on $1+2\epsilon, 1+\epsilon, \delta$such that $\frac{\delta}{k'} < \nu < 1+\epsilon+\frac{\delta}{k'}$. For $\epsilon > 0$ we have, by Holder's inequality (6.1)

$$\{\Gamma(\delta)h_\delta(1-\epsilon)\}^{(1+2\epsilon)}$$

$$\leq \left\{\int_0^{1+\epsilon}\left(\log\frac{1}{\sigma}\right)^{(1+2\epsilon)\mu}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1} h^{1+2\epsilon}(\sigma)\frac{d\sigma}{\sigma}\right\}\left\{\int_0^{1+\epsilon}\left(\log\frac{1}{\sigma}\right)^{k'\mu}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1}\frac{d\sigma}{\sigma}\right\}^{\frac{1+2\epsilon}{k'}}$$

$$= A\big((1+2\epsilon),(1+\epsilon),\delta\big)\int_0^1 (\log[1/(1-\epsilon)])^{((1+2\epsilon)\delta-(1+2\epsilon)\mu)}\int_0^{1+\epsilon} (\log[1/\sigma])^{(1+2\epsilon)\mu}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1} h^{(1+2\epsilon)}(\sigma)\,d\sigma/\sigma, \qquad (6.1)$$

The second factor on the right of the first line of (6.1) being evaluated by means of the substitution$\frac{1}{x} = \log\frac{1}{(1-\epsilon)}, \frac{1}{y} = \log\frac{1}{\sigma}$.If $\epsilon = 0$, the final inequality (6.1) holds trivially (where$\frac{1}{k'}$ is interpreted as 0).Writing $c = \frac{(1+2\epsilon)(1+\epsilon)\delta}{k'} - (1+2\epsilon)\mu$, we therefore have for $\epsilon \geq 0$,

$$\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} h_\delta^{(1+2\epsilon)}(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$$
$$\leq A(1+2\epsilon,1+\epsilon,\delta)$$
$$\cdot \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{c-1}\frac{d\epsilon}{1-\epsilon}\int_0^{1+\epsilon}\left(\log\frac{1}{\sigma}\right)^{(1+2\epsilon)\mu}\left(\log\frac{(1-\epsilon)}{\sigma}\right)^{\delta-1} h^{1+2\epsilon}(\sigma)\frac{d\sigma}{\sigma}$$
$$= A(1+2\epsilon,1+\epsilon,\delta)$$
$$\cdot \int_0^1 \left(\log\frac{1}{\sigma}\right)^{(1+2\epsilon)\mu} h^{1+2\epsilon}(\sigma)\frac{d\sigma}{\sigma}\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{c-1}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\delta-1}\frac{d(1-\epsilon)}{1-\epsilon} \quad (6.2)$$

On substituting $\delta = \frac{1}{1-\epsilon}, t = \log\frac{1}{\sigma}$ we see that the inner integral on the right of (6.2) is equal to$A(1+2\epsilon, 1+\epsilon, \delta \log 1\sigma c+\delta-1$.and the result now follows.

**7.We take next the proof of Theorem 3, and here we useTheorem F (so that the proof,like that for$g_{1+2\epsilon,1+\epsilon}$ depends ultimately on the Littlewoods– Paleyg-theorem):**

As remarked above,it is enough to prove(5.3) when $\varepsilon + 1$ is a largeInteger.We note now that if $E = \frac{1}{z-1}$ then

$$v^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\epsilon)e^{i\theta-it}\varphi'\left((1-\epsilon)e^{i\theta-it}\right)v^\epsilon E\left((1-\epsilon)e^{it}\right)dt.$$

It is immediate from(3.5)that for positive integral$(1+\epsilon)^2$

$$\left|v^\epsilon E(1-\epsilon)e^{it}\right| \leq A(1+\epsilon)(1-\epsilon)\left|1-(1-\epsilon)e^{it}\right|^{-(1-\epsilon)}, \quad (7.1)$$

And thereforealso

$$(1-\epsilon)^{-2(1+2\epsilon)}\left|v^{1+\epsilon}\left(\varphi(1-\epsilon)^2 e^{i\theta}\right)\right| \leq A(1+\epsilon)\left\{\int_{-\pi}^{\pi}\frac{\left|\varphi' i(1-\epsilon)e^{i\theta-it}\right|}{|1-(1-\epsilon)e^{it}|^{1+\epsilon}}\right\}^{1+2\epsilon}$$
$$\leq A(1+\epsilon)\left\{\int_{-\pi}^{\pi}\frac{\left|\varphi' l(1-\epsilon)e^{i\theta-it}\right|\,dt}{|1-(1-\epsilon)e^{it}|^{(1+2\epsilon)(\epsilon-1)+2}}\right\}\left\{\int_{-\pi}^{\pi}\frac{dt}{|1-(1-\epsilon)e^{it}|^2}\right\}^{2\epsilon}$$
$$= A(1+2\epsilon,1+\epsilon)(1-(1-\epsilon)^2)^{-2\epsilon}\int_{-\pi}^{\pi}\frac{\left|\varphi'\left((1-\epsilon)e^{i\theta-it}\right)\right|^{1+2\epsilon}dt}{|1-(1-\epsilon)e^{it}|^{(1+2\epsilon)(\epsilon-1)+2}}$$

**8.For the proof ofTheorems 4. We useanargumentof atypefirstemployedby Littlewoods andPaleyforthecase$\epsilon = 0$of the $g_{1+2\epsilon,1+\epsilon}$-theorems,and subsequently extended by Hirschman[18]and the author [6] to the case$\epsilon \neq 0$.For$G_{1+2\epsilon,1+\epsilon}$the argument takes a verysymmetrical form:**

To prove Theorem 4,it is enough to show that if if $0 < \epsilon \leq \frac{1}{2}$, then

$$M_{1+\epsilon}(\varphi, R) \leq A(1+2\epsilon, 1+\epsilon, 1+\epsilon)N_{1+\epsilon}\left(G_{1+2\epsilon,1+\epsilon}\right) \quad (8.1)$$

For $0 < R < 1$, since the expression on the left of (34) is equal to

$$\sup\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi\left(Re^{i\theta}\right)V(\theta)\,d\theta\right|$$

Where the supremum is taken over all complex – valued trigonometric polynomials$V$ satisfying $\mu_{\frac{1+\epsilon}{\epsilon}}(V) = 1$, it is therefore enough to prove that for any such$V$

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi\left(Re^{i\theta}\right)V(\theta)\,d\theta\right| \leq A(1+2\epsilon,1+\epsilon,1+\epsilon)\mu_{1+\epsilon}\left(G_{1+2\epsilon,1+\epsilon}\right) \quad (8.2)$$

When $0 < R < 1$.Let

$$V(\theta) = \sum_{\epsilon=N-1}^{N} k_{1+\epsilon}e^{(1+\epsilon)i\theta}, \text{let } \xi(z) = \sum_{\epsilon=0}^{N} k_{-(1+\epsilon)}z^{1+\epsilon}$$

and for any $\gamma > 0$ and $0 < R < 1$ let

$$H_{k',\gamma}(\theta) = \left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{k'\gamma-1}\left|v^\gamma\,\xi\left((1-\epsilon)e^{i\theta}\right)\right|^{k'}\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{k'}}$$

By Theorem 3 and Theorem G

$$\mu_{\frac{1+\epsilon}{\epsilon}}(H_{k',\gamma}) \le A(1+2\epsilon,1+\epsilon,\gamma)M_{\frac{1+\epsilon}{\epsilon}}(\xi,R) \le A(1+\epsilon,1+\epsilon,\gamma)\mu_{\frac{1+\epsilon}{\epsilon}}(V) = A(1+2\epsilon,1+\epsilon,\gamma) \quad (8.3)$$

We note now that

$$\int_{-\pi}^{\pi} \varphi(Re^{i\theta})V(\theta)\, d\theta = 2\pi \sum_{\epsilon=0}^{N} c_{1+\epsilon} k_{-(1+\epsilon)} R^{1+\epsilon}$$

And hence, by the (19) for any positive $1+\epsilon,\gamma$,

$$\int_{-\pi}^{\pi} \varphi(Re^{i\theta})V(\theta)\, d\theta = \frac{2^{1+\epsilon}}{\Gamma(1+\epsilon+\gamma)} \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\epsilon+\gamma} \frac{d\epsilon}{1-\epsilon} \int_{-\pi}^{\pi} v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right) v^{\gamma}\, \xi\left(R(1-\epsilon)e^{i\theta}\right) d\theta$$

$$= \frac{2^{1+\epsilon}}{\Gamma(1+\epsilon+\gamma)} \int_{-\pi}^{\pi} d\theta \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\epsilon+\gamma} v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right) v^{\gamma}\, \xi\left(R(1-\epsilon)e^{i\theta}\right) \frac{d(1-\epsilon)}{1-\epsilon} \quad (8.4)$$

By Holder's inequality with indices$1+2\epsilon, k'$ the absolute value of the inner integral on the right of (8.4)does not exceed $G_{1+2\epsilon,1+\epsilon}(\theta)H_{k',\gamma}(-\theta)$and there exceed fore, by Holder's inequality with indices$1+\epsilon, \frac{1+\epsilon}{\epsilon}$

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(Re^{i\theta})V(\theta)\, d\theta\right| \le \frac{2^{1+\epsilon+\gamma}}{2\pi\Gamma(1+\epsilon+\gamma)} \int_{-\pi}^{\pi} G_{1+2\epsilon,1+\epsilon}(\theta)H_{k',\gamma}(-\theta)$$

$$\le \frac{Z^{1+\epsilon+\gamma}}{\Gamma(1+\epsilon+\gamma)} N_{1+\epsilon}\left(G_{1+2\epsilon,1+\epsilon}\right)\mu_{\frac{1+\epsilon}{\epsilon}}\left(H_{k',\gamma}\right) \quad (8.5)$$

Taking $\gamma = 1$(say), we obtain from (8.5)and (8.4)the inequality (8.2)and this completes the proof when $\epsilon = +\infty$, the inequality (33) is false for all $\epsilon > -1$. To prove thisLet $\varphi(z) = \sum_{\epsilon=1}^{\infty}(1+\epsilon)(\log(1+\epsilon))^{-1}z^{1+\epsilon}$, so that $\varphi$ is unbounded in $\Delta$. Then for $\epsilon > -1$we have

$$\left|v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right)\right| \sum_{\epsilon=1}^{\infty}(1+\epsilon)(\log(1+\epsilon))^{-1}(1-\epsilon)^{1+\epsilon} \le A(1+\epsilon)(1+\epsilon)^2(\epsilon)^{-(1+\epsilon)}\left(\log\frac{e}{\epsilon}\right)^{-1}$$

Whence $G_{1+2\epsilon,1+\epsilon}(\theta) \le A(1+2\epsilon,1+\epsilon)$ for all $\theta$(since$\epsilon > 0$), and this prove the statement.

We note in passing that the results for the function$g_{1+2\epsilon,1+\epsilon}$definedin(5.5)correspondingtoTheorems3and 4arenowimmediateconsequences of(5.6)and(5.7).When$\epsilon > 0$we have also an inequality for$g_{1+\epsilon 2,1+\epsilon}$corresponding to(5.2),but we postpone the proof of this until §16.

**9-It is probable that the inequality of Theorem 4 holds for $\frac{1}{2} < \epsilon \le 1$.We are unable to prove this in full generality, but we can deal with the case$-\frac{1}{2} < \epsilon \le 1$for certain values of $(1+\epsilon)^3$.In contrasttoTheorem4,thecase $\epsilon = +\infty$ is true here.**

**Theorem 5.** If $\varphi(0) = 0$and either (i) $\epsilon$(ii)$0 < \epsilon \le\infty$, Then

$$\mu_{1+\epsilon}(\varphi) = A(1+2\epsilon,1+\epsilon,1+\epsilon)N_{1+\epsilon}\left(G_{1+2\epsilon,1+\epsilon}\right) \quad (9.1)$$

We consider first the case where $\varphi$ is regular in the closed disc $\overline{\Delta}$, and we show that in this case the inequality (9.1)holds for $\epsilon \ge 0, -\frac{1}{2} \le \epsilon \le 0$, the limitations on $\epsilon+1$ in (i) and (ii) arise only in the reduction of the general case to this special one. Suppose then that $\varphi$ is regular in $\overline{\Delta}$ and that$\epsilon \ge 0, -\frac{1}{2} < \epsilon \le 0$. It is enough to show that

$$M_{1+\epsilon}(\varphi,1) \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\mu_{1+\epsilon}\left(G_{1+2\epsilon,1+\epsilon}\right) \quad (9.2)$$

Since $\varphi$ is regular I n $\overline{\Delta}$, the formulae (3.7) give

$$\varphi(e^{i\theta}) = \frac{1}{\Gamma(1+\epsilon)} \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\epsilon} v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right) \frac{d\epsilon}{1-\epsilon},$$

and therefore

$$\left|\varphi(e^{i\theta})\right| \le \frac{1}{\Gamma(1+\epsilon)} \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\epsilon} \left|v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right)\right| \frac{d(1-\epsilon)}{1-\epsilon} \quad (9.3)$$

This trivially implies (9.2)for$\epsilon = 0$,. Let $\phi$ be defined as in Theorem 1 with $\epsilon = \frac{1}{2}$(say). Then

$$\left|v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right)\right| \le A(\epsilon+1)(1-\epsilon)(\epsilon)^{-(\epsilon+1)}\Phi(\theta) \le A(1+\epsilon)\left(\log\frac{1}{1-\epsilon}\right)^{-(\epsilon+1)} \Phi(\theta),$$

Whence, by (9.3)

$$\left|\varphi(e^{i\theta})\right| \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\Phi^{-2\epsilon}(\theta) \int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left((1-\epsilon)e^{i\theta}\right)\right|^{2\epsilon+1} d(1-\epsilon)$$

$$= A(1+2\epsilon,1+\epsilon,1+\epsilon)\Phi^{-2\epsilon}(\theta)G_{1+2\epsilon,1+\epsilon}^{1+2\epsilon}(\theta) \quad (9.4)$$

If$\epsilon < \infty$, then (9.4) gives

$$M_{1+\epsilon}(\varphi,1) \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\left\{\int_{-\pi}^{\pi} \Phi^{2\epsilon(1+\epsilon)}(\theta) G_{1+2\epsilon,1+\epsilon}^{(1+2\epsilon)(1+\epsilon)}(\theta)\, d\theta\right\}^{\frac{1}{1+\epsilon}}$$

Applying Holder's inequality with indices $\frac{1}{1-k}, \frac{1}{k}$ and then Theorem E, we obtain

$$M_{1+\epsilon}(\varphi,1) \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\mu_{1+\epsilon}^{2\epsilon}(\Phi)\mu_{1+\epsilon}^{2\epsilon+1}(G_{1+2\epsilon,1+\epsilon})A(1+2\epsilon,1+\epsilon,1+\epsilon)M_{1+\epsilon}^{2\epsilon}(\varphi,1)\mu_{1+\epsilon}^{2\epsilon+1}(G_{1+2\epsilon,1+\epsilon}) \quad (9.5)$$

And since $M_{1+\epsilon}(\varphi,1)$isfinite, this implies (40). If $\epsilon = +\infty$, then (9.2)follows immediately from (9.5), and again we obtain (9.4).

Suppose now that$\varphi$ is regular in$\Delta$, and let $0 < R < 1$. Aplying the special case to the function$z \to \varphi(Rz)$, we get

$$M_{1+\epsilon}^{1+\epsilon}(\varphi,R) \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\int_{-\pi}^{\pi} d\theta \left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right| \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}} \quad (9.6)$$

If $\epsilon \ge 0$, then

$$\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon} \frac{d\epsilon}{1-\epsilon} = \int_0^R \left(\log\frac{1}{\sigma}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon} \frac{d\sigma}{\sigma}$$
$$\le \int_0^R \left(\log\frac{1}{\sigma}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon} \frac{d\sigma}{\sigma} \le G_{1+2\epsilon,1+\epsilon}^{1+2\epsilon}(\theta)$$

Hence

$$M_{1+\epsilon}^{1+2\epsilon}(\varphi,1) \le A(1+2\epsilon,1+\epsilon,1+\epsilon)\mu_{1+\epsilon}^{1+\epsilon}(G_{1+2\epsilon,1+\epsilon})$$

and this implies (9.1). If $\epsilon > -1$, then (9.6)gives

$$M_{1+\epsilon}^{1+2\epsilon}(\varphi,1) \le A(1+2\epsilon,1+\epsilon)\int_0^R \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \frac{d\epsilon}{1-\epsilon}\int_{-\pi}^{\pi} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon} d\theta$$

Since the inner integral on the right increases with $R$, we may replace $R$ on the right by 1, and this again implies (9.1).

We note explicitly the case $\epsilon = 0$ of Theorem 4) and 5, viz

**Theorem 6.** If$\varphi(0) = 0$, and $-\frac{1}{2} < \epsilon \le 1$, then

$$\mu_{1+2\epsilon}(\varphi) \le A(1+2\epsilon,1+\epsilon)\left\{\int_{-\pi}^{\pi}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{1+\epsilon}\, \varphi\left(R(1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}(1-\epsilon)^{-1} d\theta\, d(1-\epsilon)\right\}^{\frac{1}{1+2\epsilon}}.$$

**10.Atheoremonthe means$M_{\epsilon+1}(\varphi \cdot 1-\epsilon)$.**We prove next

**Theorem 7.** Let $\varphi(0) = 0$, let $0 < \epsilon \le +\infty$, and let

$$J = \left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_{1+\epsilon}^{1+2\epsilon}(\varphi,1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}$$

Then

$$J \le A(1+2\epsilon,1+\epsilon,1+\epsilon)N_{1+\epsilon}(\varphi) \quad (10.1)$$

This is equivalent to a result of Hardy and Littlewood [12,Th.31;17,Th.II].The theorem can be proved in various ways, and we give here a variant of the proof in[17]which makes the least demands on the theory of the$H^{\epsilon+1}$classes.

Suppose first $-1 \le \epsilon \le 1$, and let $C = N_2(\varphi)$.

Then

$$M(\varphi,1-\epsilon) \le \sum_{\epsilon=0}^{\infty}|c_{1+\epsilon}|(1+\epsilon)^{1+\epsilon} \le \left(\sum_{\epsilon=0}^{\infty}|c_{1+\epsilon}|^2(1-\epsilon)^{1+\epsilon}\right)^{\frac{1}{2}} \le C\left(\frac{1-\epsilon}{\epsilon}\right)^{\frac{1}{2}} \le C\left(\log\frac{1}{1-\epsilon}\right)^{-\frac{1}{2}},$$

And therefore for $-\infty \le \epsilon \le 1$

$$M_{1+\epsilon}(\varphi,1-\epsilon) \le M_{\epsilon+1}^{\frac{1-\epsilon}{\epsilon+1}}(\varphi,1-\epsilon)M_2^{\frac{2}{\epsilon+1}} \le C\left(\log\frac{1}{1-\epsilon}\right)^{-(1+\epsilon)} \quad (10.2)$$

Hence

$$J^k \leq C^{2\epsilon-1} \int^1 \left(\log\frac{1}{1-\epsilon}\right)^{2\epsilon+1} M^2_{\epsilon+1}(\varphi, 1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon} \qquad (10.3)$$

Next, since

$$\varphi\left((1-\epsilon)e^{i\theta}\right) = \int_0^{1+\epsilon} v^1\, \varphi\left(\sigma e^{i\theta}\right)\frac{d\sigma}{\sigma},$$

Minkowski's inequality gives

$$M_{1+\epsilon}(\varphi, 1-\epsilon) = \int_0^{1+\epsilon} M_{1+\epsilon}(v^1\, \varphi, \sigma)\frac{d\sigma}{\sigma}$$

(the case $\epsilon = +\infty$ being included and hence, by (10.3)and the case $\epsilon = 0, \delta = 1$of Lemma 1.

$$J^{1+2\epsilon} \leq A(1+2\epsilon, 1+\epsilon)C^{2\epsilon-1}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{2\epsilon+3} M^2_{\epsilon+1}(v^1\varphi, 1-\epsilon)\frac{d\epsilon}{1-\epsilon}$$

$$= A(1+2\epsilon, 1+\epsilon)C^{2\epsilon-1}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{2\epsilon+3} M^2_{\epsilon+1}(v^1\varphi, (1-\epsilon)^2)\frac{d(1-\epsilon)}{1-\epsilon} \qquad (10.4)$$

By (10.2) applied to the function $z \to v^1\varphi\left((1-\epsilon)z\right)$

$$M^2_{\epsilon+1}(v^1\varphi, (1-\epsilon)^2) \leq \left(\log\frac{1}{1-\epsilon}\right)^{-2(\epsilon+1)} M^2_2(v^1\varphi, 1-\epsilon)$$

and hence, by (10.4)and (3.7)

$$J^{1+2\epsilon} \leq A(1+2\epsilon, \epsilon+1)C^{2\epsilon-1}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right) M^2_2(v^1\varphi, 1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$$

$$= A(1+2\epsilon, \epsilon+1)C^{2\epsilon-1}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right) \sum_{\epsilon=0}^{\infty}(1+\epsilon)^2|c_{1+\epsilon}|^2(1+\epsilon)^{2(1+\epsilon)}\frac{d(1-\epsilon)}{1-\epsilon}$$

$$= A(1+2\epsilon, \epsilon+1)C^{2\epsilon-1}\sum_{\epsilon=0}^{\infty}|c_{1+\epsilon}|^2\,(1+2\epsilon, \epsilon)C^{2\epsilon-1}\frac{d(1-\epsilon)}{1-\epsilon}$$

And this is (10.1) with $\epsilon = 1$. In this case it is enough to prove hat if $\psi$ is regular in $\Delta$, and $0 < \epsilon \leq +\infty$,, then

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M^{1+2\epsilon}_{1+\epsilon}(\psi, 1-\epsilon)(1-\epsilon)^{2\epsilon}d(1-\epsilon)\right\}^{\frac{1}{1+2\epsilon}}$$

$$\leq A(1+2\epsilon, 1+\epsilon, \epsilon+1)N_{1+\epsilon}(\psi) \qquad (10.5)$$

For the inequality (10.1) follows from this with $\psi(z) = z^{-1}\varphi(z)$. Further by Theorem C, it is enough to prove (10.5) when $\psi$ has no zeros in $\Delta$. Let $\psi$ be such a function, let $\chi = \psi^{\frac{1+\epsilon}{2}}$,$s = 2, \epsilon = -1$. Then $\delta > 2, \epsilon = 0$ and$N^{1+\epsilon}_{1+\epsilon}(\psi) = N^2_2(\chi)$, so that for this $\psi$ the inequality (10.5) is equivalent to

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M^1_s(\chi, 1-\epsilon)(1-\epsilon)^{2\epsilon}d(1-\epsilon)\right\}^{\frac{1}{1+\epsilon}}$$

$$\leq A(1+2\epsilon, 1+\epsilon, \epsilon+1)N_2(\chi) \qquad (10.6)$$

But, by the case $\epsilon = 1$ of (10.1) applied to $\varphi(z) = z\chi(z)$

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M^1_s(\chi, 1-\epsilon)(1-\epsilon)^{2\epsilon}d(1-\epsilon)\right\}^{\frac{1}{1+\epsilon}}$$

$$\leq A(1+2\epsilon, \epsilon+1)N_2(\chi) \qquad (10.7)$$

If $k \geq l$, (10.7) implies (10.6) immediately. If $\epsilon > 0$, then on putting $(1-\epsilon)^{1+\epsilon} = \sigma^{1+2\epsilon}$in the integral on the left of (10.7) and note that$M_s(\chi, 1-\epsilon) \geq M_s(\chi, \sigma)$ (since $1-\epsilon = \sigma^{\frac{1+2\epsilon}{1+\epsilon}} > \sigma$), we see that the left side of (10.6) does not exceed $\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{1+\epsilon}$ times that of (10.7), whence again (10.6) follows, and this completes the proof.

For certain $1+\epsilon$ we have a stronger result

**Theorem 8.** Let $\varphi(0) = 0$, let $w = (w_{1+\epsilon})$ be a sequence of numbers such that $|w_{1+\epsilon}| \leq 1$ for all $\epsilon + 1$, and let

$$\varphi_w(z) = \sum_{z=0}^{\infty} c_{1+\epsilon}w_{1+\epsilon}z^{1+\epsilon}\ (z \in \Delta) \qquad (10.8)$$

If $-1 < \epsilon \leq \infty$, then

$$\left[\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_{\epsilon+1}^{1+2\epsilon}(\varphi_w, 1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right]^{\frac{1}{1+2\epsilon}}$$
$$\leq A(1+2\epsilon, 1+\epsilon, \epsilon+1)N_{1+\epsilon}(\varphi) \quad (10.9)$$

If $\epsilon = 1$, this follows from the trivial $N_2(\varphi_w) \leq \mu_2(\varphi)$, and inequality (10.1) applied to $\varphi_w$.
If $\epsilon < 1$, then by (10.1) with $\epsilon = 1$ we have

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{-(2\epsilon^2+\epsilon+1)} M_2^{1+2\epsilon}(\varphi, 1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}} \leq A(1+2\epsilon, 1+\epsilon)\mu_{1+\epsilon}(\varphi) \quad (10.10)$$

Further, by (10.2) applied to $\varphi_w$, we have

$$\left(\log\frac{1}{1-\epsilon}\right)^{\frac{1+2\epsilon}{1+\epsilon}} M_{\epsilon+1}(\varphi, 1-\epsilon) \leq M_2(\varphi_w, 1-\epsilon) \leq M_2(\varphi, 1-\epsilon) \quad (10.11)$$

For $1 \leq \epsilon \leq +\infty$, and (10.10) and (10.11) together give (10.11)) choosing $w$ in Theorem 8 so that $c_{1+\epsilon}w_{1+\epsilon} = |c_{1+\epsilon}|$ for all $1+\epsilon$, we deduce the following result.

$$\varphi_*(z) = \sum_{\epsilon=0}^{\infty} |c_{1+\epsilon}| z^{1+\epsilon} \ (z \in \Delta) \quad (10.12)$$

If $-1 \leq \epsilon \leq +\infty$, then 1

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_\epsilon^{1+2\epsilon}(\varphi_*, 1-\epsilon_*)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}} \leq A(1+3\epsilon, \epsilon+1, \epsilon+1)N_{1+\epsilon}(\varphi)$$

In particular, if $-1 < \epsilon \leq 1$ then

$$\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{1+\frac{2\epsilon}{\epsilon}} \varphi_*^{1+2\epsilon}(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}} \leq A(1+2\epsilon, \epsilon+1)N_{1+\epsilon}(\varphi) (10.13)$$

And if $0 < \epsilon \leq 1$, then

$$\left\{\int_0^1 \varphi_*^{1+2\epsilon}(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+\epsilon}} \leq A(1+\epsilon)N_{1+\epsilon}(\varphi) \quad (10.14)$$

The inequality (10.14)is equivalent to a theorem of Hardy and Littlewood [8,Th.15],and(10.13) can be deduced from two results Of the same authors[10,Th.3;8,Th.5].The proofs of these results given by Hardy and Littlewood make use of the inequality

$$\left\{\sum_{\epsilon=0}^{\infty} (1+\epsilon)^{\epsilon-1}|c_{1+\epsilon}|^{1+\epsilon}\right\}^{\frac{1}{1+\epsilon}} \leq A(1+\epsilon)N_{1+\epsilon}(\varphi) \ (10.15)$$

Where $0 < \epsilon \leq 1$, and are a good deal less elementary then proof above.
It has been shown by Hardy and Littlewood [8] that for $0 < \epsilon$ the inequality (10.14) implies (10.15), the argument here being relatively simple. We thus obtain effectively a new proof of (10.15) for $-1 < \epsilon < 0$ It is natural here to ask whether

$$N_{1+\epsilon}(\varphi) \leq A(1+\epsilon)N_{1+\epsilon}(\varphi)(\epsilon > -1)m(\epsilon \neq 1) \quad (10.16)$$

For every sequence $w = (w_{1+\epsilon})$ such that $|w_{1+\epsilon}| \leq 1$. As might be expected the answer is negative. If (10.16) were true for $\epsilon > 1$, and this is known to be false a counter - example being

$$\varphi(z) = \sum (\epsilon+1)^{-\frac{1}{2}-\delta} e^{i(1+\epsilon)\log(1+\epsilon)} z^{1+\epsilon} (\delta > 0)$$

(Hardy and Littlewood [8,p.206]).This argument show also that the inequality

$$N_{1+\epsilon}(\varphi_*) \leq A(1+\epsilon)N_{1+\epsilon}(\varphi) \ (10.17)$$

is false for $\epsilon > 1$.

To disprove (10.16) for $\epsilon < 1$, we may take $\varphi(z) = \sum(1+\epsilon)^{-\frac{1}{2}} z^{1+\epsilon}, w_{1+\epsilon} = e^{I(1+\epsilon)\log(1+\epsilon)}$.
Here $\varphi \in H^{1+\epsilon}$ for $\epsilon < 1$. On the other hand $\varphi_n$ has nowhere a radial limit so that $N_{1+\epsilon}(\varphi_w) = +\infty$ for all $1+\epsilon$ (see [22,i,p.186] and [21]). The question whether (10.17) holds for $\epsilon < 1$ seems to be open (see[15]).
**11.The Hardy-Littlewood theorem on fractional integrals:** The preceding results enable us to give a succinct proof of the Hardy Littlewood theorem on fractional integrals ([12,17];see also [22,ii,p.140]).
**Theorem 9.** If $\varphi(0) = 0$ an $\epsilon > -1$ then

$$\mu_{\epsilon+1}(v_{1+\epsilon}\varphi) \leq (1+\epsilon, \epsilon+1)\mu_{1+\epsilon}(\varphi) \quad (11.1)$$

Suppose first that $\epsilon \leq 1$, and let $1+2\epsilon = \min\{\epsilon+1, 2\}$. Then by Theorem 6 and the case $\epsilon = \frac{1}{2}$ of Theorem 4, we have

$$\mu_\epsilon(\varphi) \le (\epsilon, 1+\epsilon + 1)\left\{\int_{-\pi}^{\pi} d\theta \left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|v^{\epsilon+1}\varphi\left((1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon} \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{\epsilon+1}{1+2\epsilon}}\right\}^{\frac{1}{\epsilon+1}} \quad (11.2)$$

Since $v^{1+\epsilon}(v_{1+\epsilon}\varphi) = \varphi$, applying successively (11.2) with$\varphi$ replaced by $v_{1+\epsilon}\varphi$, Minkowski's inequality, and Theorem 7, we obtain.

$$\mu_\epsilon(v_{1+\epsilon}\varphi) \le A(1+\epsilon, \epsilon+1)\left\{\int_{-\pi}^{\pi} d\theta \left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} \left|\varphi\left((1-\epsilon)e^{-\theta}\right)\right|^{1+2\epsilon} \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1+\epsilon}{1+2\epsilon}}\right\}^{\frac{1}{1+\epsilon}}$$

$$\le A(1+\epsilon, 1+\epsilon)\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_{1+\epsilon}^{1+2\epsilon}(\varphi, (1-\epsilon))\frac{d(1-\epsilon)}{1-\epsilon}\right\} \le A(1+\epsilon, 1+\epsilon)N_{1+\epsilon}(\varphi)$$

As required.

This leaves only the case$\epsilon > 1$. To deal with this, we can use a simple conjunct argument which enables us to deduce the required result from the case $1 < \epsilon$ already proved. Since the argument is a particular case.

If$\epsilon \ge 0$, the result of Theorem 9 continues to hold for $\epsilon = +\infty$. To prove this we use the case $\epsilon = +\infty$ of Theorem 5 (i) and Theorem 7. We thus obtain

$$\mu\left(v_{\frac{1}{1+\epsilon}}\varphi\right) \le A(1+\epsilon)\sup_\theta\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\frac{1}{\epsilon}} \left|\varphi\left((1-\epsilon)e^{1\theta}\right)\right|\frac{d\theta}{1-\epsilon}\right]$$

$$\le A(1-\epsilon)\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\frac{1}{\epsilon}} M(\theta, 1-\epsilon)\frac{d\epsilon}{1-\epsilon} \le A(\epsilon)\mu_{1+\epsilon}(\varphi) \quad (11.3)$$

This can be strengthened slightly, as can also the case $\epsilon \le 1$ of Theorem 9. Let $w = (w_{1+\epsilon})$ be a sequence of numbers such that $|w_{1+\epsilon}| \le 1$, and let $\varphi_w$ be defined as in (10.8). Since

$$v_{1+\epsilon}\varphi_w = v_0\varphi_w = v_{\frac{1}{2}-\frac{1}{\epsilon}}\left(v_{\frac{1}{\epsilon}}\varphi_w\right),$$

We have

$$\mu_{\epsilon+1}(v_{1+\epsilon}\varphi_w) \le A(1+\epsilon)\mu_2\left(v_{\frac{1}{\epsilon+1}-\frac{1}{2}}\varphi\right) \le A(1+\epsilon)\mu_2\left(v_{\frac{1}{\epsilon+1}-\frac{1}{2}}\varphi\right) \le A(1+\epsilon, 1+\epsilon)\mu_{+\epsilon}(\varphi)$$

By a double application of Theorem 9.In particular, if $\varphi_*$ is the majorant of $\varphi$ defined in (10.12) then

$$\mu_{+\epsilon}(v_{1+\epsilon}\varphi_*) \le A(1+\epsilon, +\epsilon)\mu_{1+\epsilon}(\varphi) \quad (11.4)$$

It follows from a theorem of Hardy and Littlewood on majorants [11] that (11.4) is stronger then (11.1) when $\epsilon$ is an even integer, and it is probably stronger for all $\epsilon \ge 1$.

If$0 < \epsilon \le +\infty$, then the argument above can be combined with that of (11.3) and (with $\varphi_w = \varphi_*$) gives the inequality.

$$\sum_{\epsilon=0}^{\infty}(1+\epsilon)^{-\frac{1}{1+\epsilon}}|c_{1+\epsilon}| \le A(1+\epsilon)\mu_{1+\epsilon}(\varphi)\ (-1 < \epsilon < 0)$$

This, however,is weaker than the case$\epsilon < 0$of (10.15) (see Hardy and Littlewood [12,p.421]).

**12.Theorem 6 enables us also to give simplified proof of the following theorem of Hardy and Littlewood [12,Th.46]:**

**Theorem 10.** Let $-1 < \epsilon \le +\infty, 0 < 1+\epsilon < \gamma$, let $\varphi(0) = 0$, and let

$$M_{1+\epsilon}(\varphi, 1-\epsilon) \le \left(\log\frac{1}{1+\epsilon}\right)^{-\gamma} \quad (-1 < \epsilon < 0)$$

Then

$$M_{1+\epsilon}(v_{1+\epsilon}\varphi, 1-\epsilon) \le A(1+\epsilon, 1+\epsilon, \gamma)\left(\log\frac{1}{1+\epsilon}\right)^{1+\epsilon-\gamma} \quad (-1 < \epsilon < 0)$$

Suppose first that $1 \le \epsilon \le +\infty$. By (3.6),

$$v_{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right) = \frac{1}{\Gamma(1+\epsilon)}\int_0^{1+\epsilon}\left(\frac{1-\epsilon}{\sigma}\right)^{\epsilon}\varphi(\sigma e^{i\theta})\frac{d\sigma}{\sigma}$$

Whence, by Minkowski's inequality,

$$M_{1+\epsilon}(v_{1+\epsilon}, 1-\epsilon) \le \frac{1}{\Gamma(+\epsilon)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\epsilon} M_{1+\epsilon}(1+\epsilon, \sigma)\frac{d\sigma}{\sigma}$$

$$\leq \frac{1}{\Gamma(1+\epsilon)}\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{\epsilon}\left(\log\frac{1}{\sigma}\right)^{-\gamma}\frac{d\sigma}{\sigma} = \frac{\Gamma\big(\gamma-(1+\epsilon)\big)}{\Gamma(\gamma)}\left(\log\frac{1}{1-\epsilon}\right)^{1+\epsilon-\gamma} \quad (12.1)$$

The last integral in (12.1) being evaluated by the substitution $\frac{1}{y}\log\frac{1}{\sigma},\frac{1}{\chi} = \log\frac{1}{1-\epsilon}$

Suppose next that $-1 < \epsilon < 0$, By Theorem 6with $\epsilon = 0$, applied to the function $z \to v_{1+\epsilon}\varphi\big((1-\epsilon)z\big)$, we have

$$M_{1+\epsilon}^{1+\epsilon}(v_{1+\epsilon}\varphi, 1-\epsilon) \leq A(1+\epsilon,1+\epsilon)\int_0^1\left(\log\frac{1}{t}\right)^{(1+\epsilon)(1+\epsilon)-1} M_{1+\epsilon}^{1+\epsilon}(\varphi,(1-\epsilon)t)\frac{dt}{t}$$

$$\leq A(1+\epsilon,1+\epsilon)\int_0^1\left(\log\frac{1}{t}\right)^{(1+\epsilon)(1+\epsilon)-1}\left(\log\frac{1}{(1+\epsilon)t}\right)^{-(1+\epsilon)\gamma}\frac{dt}{t}$$

$$= A(1+\epsilon,1+\epsilon)\int_0^{1+\epsilon}\left(\log\frac{1-\epsilon}{\sigma}\right)^{(1+\epsilon)(1+\epsilon)-1}\left(\log\frac{1}{\sigma}\right)^{-(1+\epsilon)\gamma}\frac{d\sigma}{\sigma}$$

$$= A(1+\epsilon,1+\epsilon,\gamma)\left(\log\frac{1}{1-\epsilon}\right)^{(1+\epsilon)(1+\epsilon-\gamma)}$$

(by the same substitution as before),and this completes the proof.Combining Theorem 10 with Theorem 1, Corollary 2, we obtain The following result (cf.[12,Th.46]).

**Theorem 11.** Let $-1 < \epsilon \leq +\infty, \gamma > 0, \gamma < 1+\epsilon$, let $\varphi(0) = 0$ and let

$$M_{1+\epsilon}(\varphi, 1-\epsilon) \leq \left(\log\frac{1}{1-\epsilon}\right)^{-\gamma} \quad (-1 < \varphi < 0)$$

Then

$$M_{1+\epsilon}(v_{1+\epsilon}\varphi, 1-\epsilon) \leq A(1+\epsilon,1+\epsilon,\gamma)\left(\log\frac{1}{1-\epsilon}\right)^{1+\epsilon-\gamma} \quad (-1 < \epsilon < 0)$$

**13.The convolution series of two power series:** We suppose Throughout this section that $\varphi, \psi$ are regular in $\Delta$, and that

$$\varphi(z) = \sum_{\epsilon=0}^{\infty} c_n z^n, \quad \psi(z) = \sum_{n=1}^{\infty} d_{1+\epsilon}z^{1+\epsilon}, \chi(z) = \sum_{\epsilon=0}^{\infty} c_{1+\epsilon}d_{1+\epsilon}z^{1+\epsilon}$$

It is easily verified that $\chi$ is regular in $\Delta$, and that

$$\chi\big((1-\epsilon)^2 e^{i\theta}\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi\left((1-\epsilon)e^{i\theta-it}\right)\psi\left((1-\epsilon)e^{it}\right)dt$$

It follows immediately from Theorem A that if $\epsilon \geq 0$, (so that $\max\{1+\epsilon,\} \leq \frac{1-\epsilon}{1+\epsilon} \leq +\infty$), and $\varphi \in H^{1+\epsilon}, \psi^1 \in H^{\epsilon+1}$, then $\chi \in H^{\frac{1-\epsilon}{1+\epsilon}}$.

Hardy and Littlewood [16,17]have given generalization of this result in which the condition that

$$M_{1+\epsilon}(\psi^1, 1-\epsilon) \leq K(-\epsilon)^{2\epsilon} \quad (13.1)$$

For some $\phi$, if $\epsilon = -\frac{1}{2}$,, then (13.1) is weaker than the condition that $\psi \in H^{1+\epsilon}$ (cf. Theorem 1, Corollary1), however, the conclusion that $\chi \in H^{\frac{1-\epsilon}{1+\epsilon}}$ remains valid. If $-\frac{1}{2} < \epsilon < 0$, then (13.1) is equivalent to the condition that $\psi \in Lip(1+2\epsilon, 1+\epsilon)$, and is stronger than the condition that $\psi \in H^{1+\epsilon}$. In this case the conclusion that $\chi \in H^{\frac{1-\epsilon}{1+\epsilon}}$ remains valid when $\varphi \in H^s$ for $s < 1+\epsilon$.

In this section we generalize these theorems by replacing(13.1) by a similar condition involving $M_{1+\epsilon}(v^{1+\epsilon}\psi, 1-\epsilon)$ where $\epsilon > -1$. Such results were stated by Hardy and Littlewood [16] for the case where $1+\epsilon$ is a positive integer $m$,but no proof for $m > 1$has been published. We find in fact that there are three distinct theorems

**Theorem 12.** Suppose that

$$\epsilon > -1, \epsilon \leq 1 \leq \frac{1-\epsilon}{1+\epsilon}$$

That $\varphi \in H^{1+\epsilon}$, and that

$$M_{1+\epsilon}(v^{1+\epsilon}\psi, 1-\epsilon) \leq K\left(\log\frac{1}{1-\epsilon}\right)^{-(1+\epsilon)}$$

Then

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi) \leq AK(1+\epsilon,1+\epsilon,1+\epsilon)\mu_{1+\epsilon}(\varphi)$$

In the remaining two theorems we regard $\epsilon, 1+\epsilon, \lambda$ as given, and define $r, s$ in terms of them.

**Theorem 13.** Suppose that $\epsilon \geq 0$

$$0 \le \lambda < 1+\epsilon, \frac{1}{s} = \frac{1+(1+\epsilon)^2}{1+\epsilon} - \lambda$$

(So that $0 < s < 1+\epsilon$) that $\varphi \in H^s$, and that

$$M_{1+\epsilon}(v^{1+\epsilon}\psi, 1-\epsilon) \le K\left(\log\frac{1}{1-\epsilon}\right)^{-\lambda}$$

If $\epsilon < +\infty$ (so that $\epsilon > 1$), then

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi) \le KA(1+\epsilon, 1+\epsilon, 1+\epsilon, \lambda)\mu_{1+\epsilon}(\varphi) \quad (13.2)$$

If $\epsilon < +\infty$ (so that $\epsilon = 1$), and $s \le 1$, then $\chi$ is continuous in $\bar{\Delta}$, and for each $\theta$

$$|\chi(e^{i\theta})| \le \frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{\epsilon} \left|v^{1+\epsilon}\chi\left((1+\epsilon)e^{i\theta}\right)\frac{d\epsilon}{1-\epsilon}\right.$$

$$\le A(1+\epsilon, \epsilon+1, 1+\epsilon, \lambda)M_s(\varphi) \quad (13.3)$$

**Theorem 14**. Suppose that the hypotheses of Theorem 13 hold and that in addition $s \le 2 \le 1+\epsilon$. Suppose also that $w = (w_{1+\epsilon})$ is a sequence of numbers such that $|w_{1+\epsilon}| \le 1$ for all $\epsilon + 1$ and let

$$\chi_w(z) = \sum_{\epsilon=0}^{\infty} c_{1+\epsilon}\, d_{1+\epsilon} w_{1+\epsilon} z^{1+\epsilon} (z \in \Delta)$$

If $\epsilon < +\infty$ (so that $\epsilon = 1$).Then

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi_w) \le KA(1+\epsilon, 1+\epsilon, 1+\epsilon, \lambda)\mu_s(\varphi)$$

And, in particular if

$$\chi_*(z) = \sum_{\epsilon=0}^{\infty} |c_{1+\epsilon} d_{1+\epsilon}| z^{1+\epsilon}\ (z \in \Delta)$$

Then

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi_*) \le KA(1+\epsilon, 1+\epsilon, 1+\epsilon, \lambda)\mu_s(\varphi) \quad (13.4)$$

If $\epsilon = +\infty$ (so that $\epsilon = 1$) and $s \le 1$,
Then

$$\sum_{\epsilon=0}^{\infty} |c_{1+\epsilon} d_{1+\epsilon}| \le KA(1+\epsilon, 1+\epsilon, 1+\epsilon, \lambda)\mu_s(\varphi) \ (13.5)$$

Proofs of the cases $\epsilon = -1, \epsilon < \infty$ of Theorems 12 and 13 are given by Hardy and Littlewood in [16,17].They have also proved in[13, 14] the cases $\epsilon = 0$ of the inequalities (13.3) and (13.5).The proofs of Theorems 12and13given here are similar in principle to those of the cases $\epsilon = 0$ in [17], but we have made some simplications.In the proofs of Theorems 12-14we may assume that $\epsilon = 0$, and in Theorem12 we may assume. $\epsilon = -1$ we let that $\epsilon > 0$, $0 \le \lambda \le 1+\epsilon$, and

$$M_{1+\epsilon}(v^{1+\epsilon}\psi, 1-\epsilon) \le \left(\log\frac{1}{1-\epsilon}\right)^{-\lambda}$$

We observe now that, by Parseval's theorem for any real $\gamma$ we have

$$v^{1+\epsilon+\gamma}\chi\left((1-\epsilon)^2 e^{i\theta}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left((1-\epsilon)e^{(i\theta-it)}\right) v^{1+\epsilon}\psi\left((1-\epsilon)e^{it}\right)dt$$

And hence, by Theorem (6.2.1)

$$M_\gamma(v^{1+\epsilon+\gamma}\chi, (1-\epsilon)^2) \le M_{1+\epsilon}(v^\gamma\varphi, 1-\epsilon)M_{1+\epsilon}(v^{1+\epsilon}\psi, 1-\epsilon)$$

$$\le \left(\log\frac{1}{1-\epsilon}\right)^{-\gamma} M_{\epsilon+1}(v^\gamma\varphi, 1-\epsilon) (13.6)$$

Consider first the proof of Theorem 12.Here $\epsilon \le +\infty$ and $\lambda = 1+\epsilon$, and we choose $\gamma$ in (13.6) to the fixed positive number (e · g · $\gamma = 1$). Applying successively Theorem 4 with $\epsilon = \frac{1}{2}$ and $1+\epsilon$ replaced by $1+\epsilon+\gamma$. Minkowski's inequality, the inequality (13.6) Minkowski's inequality again and Theorem 3 we obtain

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(x) \le B\left\{\int_{\pi}^{\pi} d\theta \left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{2(1+\epsilon)+\gamma-1} \left|v^{1+\epsilon}\chi\left((1-\epsilon)e^{i\theta}\right)\right|^2 \left|\frac{d(1-\epsilon)}{1-\epsilon}\right|\right\}^{\frac{2(1+\epsilon)}{1-\epsilon}}\right\}^{\frac{1-\epsilon}{1+\epsilon}}$$

$$\le B\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{2(1+\epsilon)+2\gamma-1} M^2_{\frac{1+\epsilon}{1-\epsilon}}(v^{1+\epsilon+\gamma}\chi, 1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{2}}$$

$$= B\left\{\int_0^1 \left(\log\frac{1}{1-\epsilon}\right)^{2(1+\epsilon)+2\gamma-1} M^2_{\frac{1+\epsilon}{1-\epsilon}}(v^{1+\epsilon+\gamma}\chi, (1-\epsilon)^2)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{2}}$$

$$\leq B\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{2\gamma+1} M_{1+\epsilon}^2(v^\gamma\varphi,1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{2}}$$

$$\leq B\left\{\int_{-\pi}^{\pi}d\theta\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{2\gamma+1}\left|v^\gamma\varphi\left((1-\epsilon)e^{i\theta}\right)\right|^2\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1+\epsilon}{2}}\right\}^{\frac{1}{1+\epsilon}}\leq B\mu_{1+\epsilon}(\varphi)$$

And this is the required result. We prove next the case $\delta > 2, \epsilon < +\infty$ of Theorem13. Let $1+2\epsilon = \min\left\{\frac{1+\epsilon}{1-\epsilon}, 2\right\}$. Then by theorem 6 and case $\epsilon = \frac{1}{2}$ of Theorem 4

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi) = B\left\{\int_{-\pi}^{\pi}d\theta\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1}\left|\left|v^{1+\epsilon}\chi\left((1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{(1+2\epsilon)(1+\epsilon)}{1-\epsilon}}\right\}^{\frac{1-\epsilon}{1+\epsilon}}$$

Applying successively Minkowski's inequality,the inequality(13.6) with $\gamma = 0$, and Theorem7,we obtain

$$\mu_{\frac{1+\epsilon}{1-\epsilon}}(\chi) \leq B\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_{\frac{1+\epsilon}{1-\epsilon}}^{1+2\epsilon}(v^{1+\epsilon}\chi,1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}$$

$$= B\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1} M_{\frac{1+\epsilon}{1-\epsilon}}^{1+2\epsilon}(v^{1+\epsilon}\chi,(1-\epsilon)^2)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}$$

$$\leq B\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-(1+2\epsilon)\lambda-1} M_{\epsilon+1}^{1+2\epsilon}(\varphi,1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\frac{1}{1+2\epsilon}}\leq B\mu_s(\varphi),$$

Since $0 < s < 1+2\epsilon, 1+\epsilon-\lambda = \frac{1}{s}-\frac{1}{1+\epsilon}$. This proves the appropriate part of Theorem 13. Similarly, by applying (13,7) to $\chi_w$ and using Theorem 8 in place of Theorem 7, we obtain the case $s \leq 1, \epsilon < +\infty$ of Theorem 13,we note that if $0 \leq R < 1$, then

$$\chi\left(Re^{i\theta}\right) = \frac{1}{\Gamma(1+\epsilon)}\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{\epsilon} v^{1+\epsilon}\chi\left(R(1-\epsilon)e^{i\theta}\right)\frac{d(1-\epsilon)}{1-\epsilon} \tag{13.8}$$

Let $0 < s < 1$. Then by the increasing property of $M$ and the inequality (13.6) with $\gamma = 0$

$$= 2^{1+\epsilon}\int_{\sqrt{\delta}}^1\left(\log\frac{1}{\epsilon}\right)^{\epsilon} M(v^{1+\epsilon}\chi,(1-\epsilon)^2)\frac{d(1-\epsilon)}{\epsilon} \leq B\int_{\sqrt{\delta}}^1\left(\log\frac{1}{1-\epsilon}\right)^{\delta-\lambda} M_{1+\epsilon}(\varphi,1-\epsilon)\frac{d(1-\epsilon)}{\epsilon}$$

Further, by Theorem 7

$$\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{\epsilon-\lambda} M_{1+\epsilon}(\varphi,1-\epsilon)\frac{d(1-\epsilon)}{\epsilon}\leq BN_s(\varphi)$$

It follows that the integral on the right of (13.8) is convergent, uniformly in $(R,\theta)$, and this implies that $\chi$ is continuous in $\bar{\Delta}$, and that (13.3) holds. A similar argument, using Theorem 8 in place of Theorem 7, gives the corresponding case of Theorem 14. The reremains the case $s > 2$ of Theorem 13,which is deduced by a conjugacy argument from the case already proved.The argument here is identical to that used by Hardy and Littlewood in their proof for the case $\epsilon = 0$ but since the proof is short, we give it for the sake of completeness.

Let $s > 2$, so that also $\frac{1+\epsilon}{1-\epsilon} \geq 1+\epsilon > s > 2$. As in the proof of Theorem 4, it is enough to prove that if $V$ is a trigonometric polynomial satisfying $N_{\frac{1-\epsilon}{1+\epsilon}}(V) = 1$, then for $0 < R < 1$.

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\chi\left(R^2e^{i\theta}\right)V(\theta)d\theta\right| \leq BN_s(\varphi) \tag{13.9}$$

Let $V(\theta) = \sum_{1+\epsilon=N}^{N}\kappa_{1+\epsilon}e^{(1+\epsilon)\theta}$, and let $\xi(z) = \sum_{\epsilon=0}^{N}\kappa_{-(1+\epsilon)}z^{1+\epsilon}$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi\left(R^2e^{i\theta}\right)V(\theta)d\theta = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}V(\theta)d\theta\int_{-\pi}^{\pi}\varphi(Re^{it})\psi\left(Re^{i\theta-it}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi\left(Re^{it}\right)\xi(Re^{-it})dt \tag{13.10}$$

Where

$$\xi(Re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}V(\theta)\,\psi\left(Re^{i\theta+it}\right)dt = \sum_{\epsilon=0}^{N}\kappa_{-(1+\epsilon)}d_{1+\epsilon}\,e^{(1+\epsilon)it}$$

Hence $\zeta,\psi,\xi$ are related as $\varphi,\psi,\chi$. Since also $1 < \frac{1-\epsilon}{1+\epsilon} < s' < 2$ and $\frac{1-\epsilon}{1+\epsilon}-\frac{1}{s'} = \frac{1}{s}-\frac{1+\epsilon}{1-\epsilon}$

We may apply the case of Theorem 13 already proved to $\xi, \psi, \epsilon$ with $\frac{1-\epsilon}{1+\epsilon}, s'$ in place of $\frac{1+\epsilon}{1-\epsilon}, s$. Using also Theorem G we thus obtain

$$\mu_{s'}(\xi) \leq BN_{\frac{1-\epsilon}{1+\epsilon}}(\xi) \leq BN_{\frac{1-\epsilon}{1+\epsilon}}(V) = B \quad (13.11)$$

Applying Holder's inequality with indices $s, s'$ to integral on the right of (13.10) and using (13.11), we obtain (13.9), and this completes the proof.

**14. An alternative definition of fractional integral and derivative:** An alternative definition of fractional integral which has been used by a number of authors is as follows.As before, let $\varphi$ be regular in $\Delta$, and let

$$\varphi(z) = \sum_{\epsilon=-1}^{\infty} c_{1+\epsilon} z^{1+\epsilon} \; (z \in \Delta)$$

Then for any $\epsilon \geq -1$ we define the fractional integral $D_{1+\epsilon}\varphi$ of $\varphi$ of order $\varphi$ by

$$D_{1+\epsilon}\varphi(z) = Z^{1+\epsilon} \sum_{\epsilon=-1}^{\infty} \frac{\Gamma(\epsilon+2)}{\Gamma(2\epsilon+3)} c_{1+\epsilon} z^{1+\epsilon} = \sum_{\epsilon=-1}^{\infty} \frac{\Gamma(\epsilon+2)}{\Gamma(2\epsilon+3)} c_{1+\epsilon} z^{2\epsilon+2}, \quad (14.1)$$

Where $Z^{1+\epsilon}$ has its principal value.i.e

$$Z^{1+\epsilon} = \exp\big(1+\epsilon(\log|z| + i\arg z)\big), -\pi < \arg z \leq \pi$$

This definition is also due to Hadamard [7].By term-by-term integration, we have

$$D_{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right) = \frac{e^{(1+\epsilon)i\theta}}{\Gamma(1+\epsilon)} \int_0^{1+\epsilon} \big((1-\epsilon)-\sigma\big)^{\epsilon} \varphi\left(\sigma e^{i\theta}\right) d\sigma,$$

Where $e^{(1+\epsilon)i\theta}$ has its principal value

The definition of the fractional derivative $D^{1+\epsilon}\varphi$ if order $\epsilon \geq -1$ normally associated with the definition (14.1) is that

$$D^{1+\epsilon}\varphi(z) = \left(\frac{d}{dz}\right)^m D_{m-(1+\epsilon)}\varphi(z), \quad (14.2)$$

Where $m = [1+\epsilon] + 1$ (see Hadamard [7, p.156]).With this definition we have the series expansion

$$D^{1+\epsilon}\varphi(z) = \sum_{\epsilon=-1}^{\infty} \frac{\Gamma(\epsilon+2)}{\Gamma(1)} C_{1+\epsilon}, \quad (14.3)$$

Where $Z^{-(1+\epsilon)}$ has its principal value, and $\frac{1}{\Gamma(1)}$ is interpreted as 0 when $1+\epsilon$ is an integer $v \geq \epsilon + 2$. When $1+\epsilon$ is appositive integer,. $D^{1+\epsilon}\varphi$ is the $1+\epsilon$ the derivative of $\varphi$ in the ordinary sense the definition (14.2) is satisfactory for $-1 < \epsilon < 0$ but is less Satisfactory for non integral $\epsilon > 0$. In particular, the function $D^{1+\epsilon}\varphi$ defined above is,for some purposes,too large in the neighborhood of the origin when. $\epsilon > 0$.In the sequel we use another definition which avoids these difficulties. For $-1 \leq \epsilon < 0$ we define $D^{1+\epsilon}\varphi$ by the series (14.3), and then for $\epsilon \geq 0$ we define $D^{1+\epsilon}\varphi$ by the relation.

$$D^{1+\epsilon}\varphi(z) = D^{1+\epsilon-[1+\epsilon]} \left(\frac{d}{dz}\right)^{[1+\epsilon]} \varphi(z). \quad (14.4)$$

With this definition,we have the series expansion

$$D^{1+\epsilon}\varphi(z) = \sum_{1+\epsilon=[1+\epsilon]}^{\infty} \frac{\Gamma(\epsilon+2)}{\Gamma(1)} C_{1+\epsilon}$$

For any $\epsilon \geq -1$, where $Z^{-(1+\epsilon)}$ has its principal value. Further, if $Z = (1-\epsilon)e^{i\theta}$ and $\gamma > 1+\epsilon \geq 0$ then

$$D^{1+\epsilon}\varphi(z) = \sum_{1+\epsilon=[1+\epsilon]}^{[\gamma]-1} \frac{\Gamma(\epsilon+2)}{\Gamma(1)} C_{1+\epsilon} + \frac{e^{(\gamma-(1+\epsilon))i\theta}}{\Gamma\big(\gamma-(1+\epsilon)\big)} \int_0^{1+\epsilon} (1-\epsilon-\sigma)^{\gamma-\epsilon-2} D^{\gamma}\varphi\big(\sigma e^{i\theta}\big) d\sigma, \quad (14.5)$$

Where $e^{(\gamma-\epsilon-1)}$ has its principal value. When β is appositive integer, $D^{1+\epsilon}\varphi$ is the $\beta th$ derivative of $\varphi$ in the ordinary sense, so that in this case the definitions (14.2) and (14.4) agree. The analogue of Theorem 1for the derivative $D^{1+\epsilon}\varphi$ is as follows.

**THEOREM 15.** If $\Phi$ is defined as in Theorem 1, then for $\epsilon > 0$.

$$\left|D^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)\right| \leq A(1+\epsilon,\eta)(1-\epsilon)^{[1+\epsilon]-1+\epsilon}(\epsilon)^{-(1+\epsilon)}\Phi(\theta), (-1 < \epsilon < 0)$$

The proof is similar to that of Theorem1,and we omit it.

**15. The function associated with the derivative $D^{1+\epsilon}$ corresponding to the function $G_{1+2\epsilon,1+\epsilon}$ is defined by**

$$G_{1+2\epsilon,1+\epsilon}(\theta) = \left\{\int_0^1 (\epsilon)^{(1+2\epsilon)(1+\epsilon)-1} (1-\epsilon)^{(2\epsilon)(1+\epsilon-[1+\epsilon])} \left|D^{1+\epsilon}\varphi(1-\epsilon)e^{i\theta}\right|^{1+2\epsilon} d(1-\epsilon)\right\}^{\frac{1}{1+2\epsilon}}$$

Here it is necessary to insert some power of $1-\epsilon$ in the integral to ensure the convergence of the integral at 0 when $1+\epsilon-[1+\epsilon] \geq \frac{1}{1+2\epsilon}$. The particular choice of the power made here enables us to carry over to $G_{1+2\epsilon,1+\epsilon}$ the argument of Theorem 5, using Theorem15 in place of Theorem1.The function $G_{1,2}$ is precisely the Littlewood- Paley g-function.

The analogue of Theorem 2 for $G_{1+2\epsilon,1+\epsilon}$ is more difficult than Theorem 2 itself, and we confine our selves here to the case$\epsilon = 0$.

**THEOREM 16.**$\epsilon \geq 0$ and $\gamma > \epsilon+1 > 0$ then for each $\theta$

$$G_{1+2\epsilon,1+\epsilon} \leq A(1+2\epsilon,1+\epsilon,\gamma)\left\{\sum_{\epsilon+1=[\epsilon+1]}^{[\gamma]-1} |C_{1+\epsilon}| + G_{1+2\epsilon}(\theta)\right\} \qquad (15.1)$$

The proof of Theorem 16 depends on the following lemma.

**Lemma 2.** Let $\epsilon > 0, y > 0$, and let

$$I = \int_0^1 (x+y)^{-(c+2\epsilon)} x^{c+\epsilon-1}(1-x)^{c-1}dx$$

Then

$$I \leq A(c+2\epsilon, c+\epsilon, c) y^{-\epsilon}(1+y)^{-(c+\epsilon)} \qquad (15.2)$$

Let $B$denote a constant depending on some or all of a, b, c.If $y > \frac{1}{2}$, and $y \leq x+y \leq 3y$, whence

$$I \leq By^{-(c+2\epsilon)} \int_0^1 x^{c+\epsilon-1}(1-x)^{c-1}\,dx = By^{-(c+2\epsilon)},$$

And this trivially implies (15.2).We may therefore suppose that $0 < y \leq \frac{1}{2}$ and here it is enough to prove that $I \leq By^{-(c+\epsilon)}$. Write

$$I = \int_0^y + \int_y^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 = I_1 + I_2 + I_3.$$

In $I_{1,x\leq x+y\leq 2y}$ and $(1-x)^{c-1} \leq B$, whence

$$I_1 \leq By^{-(c+2\epsilon)} \int_0^1 x^{c+\epsilon-1}\,dx \leq By^{-\epsilon}.$$

In $I_{2,x\leq x+y\leq 2x}$ and $(1-x)^{c-1} \leq B$, whence

$$I_2 \leq B \int_y^{\frac{1}{2}} x^{-\epsilon-1}\,dx \leq By^{-\epsilon}.$$

In $I_{3,\frac{1}{2}\leq x+y\leq 2x}$ and $(1-x)^{c-1} \leq B$, whence

$$I_3 \leq B \int_{\frac{1}{2}}^1 (1-x)^{\epsilon-1}\,dx = B.$$

Hence $I \leq By^{-\epsilon} + B \leq By^{-\epsilon}$, as required when $\epsilon = 0$, the integral I can be evalued explicitly, viz

$$I = \frac{\Gamma(c+2\epsilon)\Gamma(c)}{\Gamma(\epsilon+2c)} y^{-\epsilon}(1+y)^{-(c+\epsilon)}$$

(see,for example, [1,i,p.10, formula(11)]).

We actually use two in equalities derived from Lemma 2 by simple changes of the variable,namely that if $\epsilon > 0$,, then for $-1 < \epsilon < 0$

$$\int_0^1 (1-\sigma)^{-(c+2\epsilon)}(1-\epsilon-\sigma)^{c+\epsilon-1}(\sigma)^{c-1}d\sigma \leq A(c+2\epsilon,c+\epsilon,c)(1-\epsilon)^{\epsilon+2c-1}(\epsilon) \quad (15.3)$$

And for $0 < \sigma < 1$

$$\int_0^1 (1-\epsilon)^{-(c+\epsilon)}(1-\epsilon-\sigma)^{c+\epsilon-1}(\epsilon)^{c-1}d\epsilon \leq A(c++2\epsilon,c+\epsilon,c)(1-\sigma)^{\epsilon+2c-1}\sigma^{-\epsilon} \quad (15.4)$$

The next lemma is essentially an extension of the case $\epsilon = 0$Theorem $B$.

**Lemma 3:** Let $h(\epsilon-1) > 0$ for $0 < \epsilon < 1$, and let

$$H_\delta(1-\epsilon) = \frac{1}{\Gamma(\delta)} \int_0^{1+\epsilon} (1-\epsilon-\sigma)^\delta h(\sigma)d\sigma.$$

If $\epsilon \geq 0, \delta > 0, 1-\epsilon < \frac{1}{k}$, then

$$\int_0^1 (\epsilon)^{(1+2\epsilon)(1+\epsilon)-1}(c+2\epsilon,c+\epsilon,c)^{(1+2\epsilon)(1-\epsilon)-(1+2\epsilon)\delta} H_\delta^{(1+2\epsilon)}(1-\epsilon)d\epsilon$$

$$\leq A(1+2\epsilon,1+\epsilon,\delta,1-\epsilon)\int_0^1 (\epsilon)^{(1+2\epsilon)(1-\epsilon)+(1+2\epsilon)\delta-1}(1-\epsilon)^{(1+2\epsilon)(1-\epsilon)}h^{(1+2\epsilon)}(1-\epsilon)d\epsilon \quad (15.5)$$

Choose $\mu, w$ depending on $1+2\epsilon,\delta,1-\epsilon$, such that

$$\frac{\delta}{k'}<\mu<1+\epsilon+\frac{\delta}{k'},1-\epsilon<w<\frac{1}{k'}.$$

Writing B for a constant depending on some or all of $1+2\epsilon,\delta,1-\epsilon$,obtain from Holder's inequality and (15.3) that for $\epsilon>0$

$\{\Gamma(\delta)H_\delta(1-\epsilon)\}^{1+2\epsilon}$

$$\leq\left\{\int_0^{1+\epsilon}(1-\sigma)^{(1+\epsilon)\mu}(1-\epsilon-\sigma)^{\delta-1}\sigma^{(1+2\epsilon)\omega}h^{1+2\epsilon}(\sigma)d\sigma\right\}\left\{\int_0^{1-\epsilon}(1-\sigma)^{-k'\mu}(1-\epsilon-\sigma)^{\delta-1}\sigma^{-k'\omega}d\sigma\right\}^{\frac{1+2\epsilon}{k'}}$$

$$\leq B(1-\epsilon)^{\frac{(1+2\epsilon)\delta}{k'-(1+2\epsilon)\omega}}(\epsilon)^{\frac{(1+\epsilon)\delta}{k'-(1+2\epsilon)\mu}}\int_0^1(1-\sigma)^{(1+2\epsilon)\mu}(1-\epsilon-\sigma)^{\delta-1}\sigma^{(1+2\epsilon)\omega}h^{(1+2\epsilon)}(\sigma)d\sigma, \quad (15.6)$$

Since $\mu>\frac{\delta}{k'},\delta>0$ and $\omega<\frac{1}{k'}$. If $\epsilon=0$ the final inequality in (15.6) holds trivially (where $\frac{1}{k'}$ interpreted as 0). Writing

$$c+2=\delta+(1+2\epsilon)\omega-(1+2\epsilon)(1-\epsilon),c=(1+2\epsilon)(1+\epsilon)+\frac{(1+2\epsilon)\delta}{k'-(1+2\epsilon)\mu}$$

We therefore obtain from(15.6)and(15.4)that for $\epsilon\geq 0$

$$\int_0^1(\epsilon)^{(1+2\epsilon)(1+\epsilon)-1}(1-\epsilon)^{(1+2\epsilon)\eta-(1+2\epsilon)\delta}H_\delta^{(1+2\epsilon)}(1-\epsilon)d(1-\epsilon)$$

$$\leq B\int_0^1(\epsilon)^{\epsilon-1}(1-\epsilon)^{-(c+\epsilon)}d\epsilon\int_0^{1-\epsilon}(1-\sigma)^{(1+2\epsilon)\mu}(1-\epsilon-\sigma)^{\delta-1}h^{1+2\epsilon}(\sigma)d\sigma$$

$$\leq B\int_0^1(1-\sigma)^{(1+2\epsilon)\mu}\sigma^{(1+2\epsilon)\omega}h^{1+2\epsilon}(\sigma)d\sigma\int_0^1(1-\epsilon)^{-(c+\epsilon)}(1-\epsilon-\sigma)^{\delta-1}\epsilon^{\epsilon-1}d(1-\epsilon)$$

$$\leq B\int_0^1(1-\sigma)^{(1+2\epsilon)\mu+\delta+c-1}\sigma^{(1+2\epsilon)\omega+\delta-c-2\epsilon}h^{1+2\epsilon}(\sigma)d\sigma,$$

Since $\epsilon>0,\delta>0,c>0$, and this is the required inequality.The relation of Lemma 3to the case $\epsilon=0$ of Theorem B can be seen by substituting

$$f(x)=(1+x)^{-1-\delta}h\left(\frac{x}{(1+x)}\right),(1+2\epsilon)\eta=-1-(1+2\epsilon)\lambda,(1+2\epsilon)\xi=-(1+2\epsilon)(\epsilon-\lambda)-1.$$

We thus obtain

$$\int_0^{+\infty}(1+x)^{(1+\epsilon)}x^{-1(1+\epsilon)(\lambda+\delta)}F_\delta^{1+2\epsilon}(x)\,dx$$

$$\leq A(1+2\epsilon,\delta,\lambda,\delta)\int_0^{+\infty}(1+x)^{(1+2\epsilon)\xi}x^{-1-(1+2\epsilon)\lambda}f^{1+2\epsilon}(x)dx, \quad (15.7)$$

Where $\epsilon\geq 0,\delta>0,\lambda>-1,\xi<1+\lambda$ and $F_\delta$ is defined as in Theorem B. For $\xi\leq 0$ this is an immediate consequence of Theorem B with $f(x)$ replaced by $(1+x)^\xi f(x)$ but for $0<\xi<1+\lambda$ it requires.An independent proof.There is presumably an extension of (15.7) with index $\epsilon\leq 0$ on the requires an independent proof. There is presumably an extension of (15.7) with index $\epsilon=0$ on the right,but we do not pursue this point.

Lemma 3 does not apply if $\epsilon=0,\epsilon=1$ and here we have an Almost trivial result, namely.

**LEMMA4.** Let $h,H_\delta$ be as in Lemma 3and let $\epsilon>-1,\delta>0$. Then

$$\int_0^1(\epsilon)^\epsilon H_\delta(1-\epsilon)d\epsilon\leq A(1+\epsilon,\delta)\int_0^1(\epsilon)^{\epsilon+\delta}h(1-\epsilon)d\epsilon.$$

Consider now the proof of Theorem 16.In view of the definition (14.4),it is enough to prove the inequality (15.1) when $\epsilon\geq 0$ and $[1+\epsilon]\leq 1+\epsilon\leq\gamma\leq[1+\epsilon]+1$. It is therefore enough to prove it when $[\gamma]=1+\epsilon$ and when $\gamma=[1+\epsilon]+1$.

If$[\gamma]=[1+\epsilon]$, then, by (14.5),

$$\left|D^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)\right|\leq\frac{1}{\Gamma(\gamma-1-\epsilon)}\int_0^{1-\epsilon}(1-\epsilon-\sigma)^{\gamma-1-\epsilon-1}\left|D^\gamma\varphi\left(\sigma e^{i\theta}\right)\right|d\sigma.$$

For $\epsilon=0$ we have only to apply Lemma 4 with $\delta=\gamma-1-\epsilon,\eta=(\gamma-[\gamma])\left(1-\frac{1}{1+2\epsilon}\right)$, this gives

$$\int_0^1(\epsilon)^{(1+2\epsilon)(1+\epsilon)-1}(1-\epsilon)^{(2\epsilon)(1+\epsilon-[1+\epsilon])-\delta}\left|D^{1+\epsilon}\varphi\left((1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}d(1-\epsilon)$$

$$\leq A(1+2\epsilon,1+\epsilon,\gamma)\int_0^1 (\epsilon)^{(1+2\epsilon)\gamma-1}(1-\epsilon)^{(2\epsilon)(\gamma-[\gamma])}\left|D^\gamma\varphi\left((1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}d\epsilon$$

And this obviously implies(15.1)for this case. If $\gamma=[1+\epsilon]+1$. Then (14.5) gives

$$\left|D^{1+\epsilon}\varphi(0(1-\epsilon)e^{i\theta}\right|$$
$$\leq A(1+\epsilon)\left|c_{[1+\epsilon]}\right|(1+\epsilon)^{-(1+\epsilon-[1+\epsilon])}$$
$$+\frac{1}{\Gamma(\gamma-(1+\epsilon))}\int_0^{1-\epsilon}(1-\epsilon-\sigma)^{\gamma-1-\epsilon-1}\left|D^\gamma\varphi(\sigma e^{i\theta})\right|d\sigma.$$

Here we have only to apply Lemma 3 with $\epsilon=1$ and again we obtain the required result.

**16. Lemma 3 enables us also to prove a theorem similar to Theorem16 for the function $g_{1+2\epsilon,1+\epsilon}$ defined by (5.5)**

**THEOREM 17.** If $\epsilon>0,\gamma>\epsilon+1$, then for each $\theta$.

$$g_{1+2\epsilon,1+\epsilon}(\theta)\leq A(1+2\epsilon,1+\epsilon,\gamma)g_{1+2\epsilon,\gamma}(\theta) \qquad (16.1)$$

Let $\delta=\gamma-1-\epsilon$, It is clearly enough to prove (16.1) when $\delta\geq 1$. Since$V^{1+\epsilon}\varphi=V_\delta(V^\gamma\varphi)$, we then have (exactly as in the proof of Theorem 1)

$$(1-\epsilon)^{\delta-1}\left|v^{1+\epsilon}\varphi(1-\epsilon)e^{i\theta}\right|\leq\frac{1}{\Gamma(\delta)}\int_0^{1-\epsilon}(1-\epsilon-\sigma)^\delta\sigma^{-1}\left|v^\gamma\varphi(\sigma e^{i\theta})\right|d\sigma.$$

Applying now Lemma 3 with $\epsilon=1$, we obtain (16.1).

**17-In view of Theorem 16, the argument of § 7can be applied to $G_{1+2\epsilon,1+\epsilon}$ and gives a result corresponding to Theorem 3.However,we can cover a number of such cases by using Theorem 3 directly,and we conclude with a proof of this. There are similar analogues of Theorems 4,9 and 10.**

**THEOREM 18.** Let $\epsilon>0,\epsilon>-1,\mu=\max\left\{0,\frac{1}{\epsilon}-1\right\}$, and let $(d_{1+\epsilon})$be a sequence of numbers such that, as $\epsilon\to\infty$

$$d_{1+\epsilon}=(1+\epsilon)^{1+\epsilon}\sum_{v=0}^{m}\alpha_v(1+\epsilon)^{-v}+o((1+\epsilon)^{1+\epsilon-m-1}), \qquad (17.1)$$

Where $m$ is affixed integer such that $m>\mu$ and $\alpha_0,\ldots,\alpha_m$ are fixed numbers.Let also $\varphi$ be defined as usual, let $\chi(z)=\sum_{\epsilon=0}^{\infty}c_{1+\epsilon}d_{1+\epsilon}z^{1+\epsilon}$ and let

$$\Gamma_{1+2\epsilon,1+\epsilon}(\theta)=\left\{\int_0^1\left(\log\frac{1}{1-\epsilon}\right)^{(1+2\epsilon)(1+\epsilon)-1}c(1-\epsilon)\left|\chi\left((1-\epsilon)e^{i\theta}\right)\right|^{1+2\epsilon}\frac{d(1-\epsilon)}{\epsilon}\right\}^{\frac{1}{1+2\epsilon}}, (17.2)$$

Where $c$ is bounded on the interval $[\delta,1]$ for $0<\delta<1$ and the integral in (17.2) is convergent at 0.Then for $\epsilon\geq\frac{1}{2}$

$$\mu_{\epsilon+1}\left(\Gamma_{1+2\epsilon,1+\epsilon}\right)\leq A(1+2\epsilon,\epsilon+1,1+\epsilon)\mu_{\epsilon+1}(\varphi)$$

We may obviously suppose that $\varphi(0)=C_0=0$. Let $C=\mu_{\epsilon+1}(\varphi)$, let B denote a constant depending on some or all of $1+2\epsilon,\epsilon+1$, and write

$$\Gamma_{1+2\epsilon,1+\epsilon}^{(1+2\epsilon)}=\int_{\frac{1}{2}}^{\frac{1}{2}}+\int_{\frac{1}{2}}^{1}=J_1+J_2$$

By Theorem $D,|C_{1+\epsilon}|\leq BC_{(1+\epsilon)\mu}$, and since $|d_{1+\epsilon}|\leq A(1+\epsilon)^{1+\epsilon}$, it follows that $\left|\chi\left((1-\epsilon)e^{i\theta}\right)\right|\leq BC(1-\epsilon)$ for $\frac{1}{2}\leq\epsilon\leq 1$, whence also $J_1\leq BC^{1+2\epsilon}$.

Next, by (17.1), we can write

$$\chi=\sum_{v=0}^{m}(1+\epsilon)_v\,\vartheta^{1+\epsilon}(\vartheta_v)+\xi$$

Here

$$\left|\xi\left((1-\epsilon)e^{i\theta}\right)\right|\leq A\sum_{\epsilon=0}^{\infty}(1+\epsilon)^{1+2\epsilon-m-1}|C_{1+\epsilon}|(1-\epsilon)^{1+\epsilon}\leq BC\sum_{\epsilon=0}^{\infty}(1+\epsilon)^{\epsilon+\mu-m}(1-\epsilon)^{c+\mu-m}(1-\epsilon)^{1+\epsilon}$$
$$\leq BC(\epsilon)^{-1}$$

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