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General Theorems And Functions Of Manifolds With Connections

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ABSTRACT: *The important notions of Riemannian geometry are based on Manifolds. The calculus of manifold just as topology is based on continuity, so the theory of manifolds is based on smoothness. According to the theory, the universe is smooth manifold equipped with Pseudo–Riemannian geometry which described the curvature of space time understanding this curvature is essential for the positioning of satellites into orbit around the earth. In generally Manifolds are simplifications of our familiar ideas about curves and surfaces to arbitrary dimensional objects. It is to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. In this paper we study about the general theorems of manifolds of different functions with the connections of various aspects describes possibly with the affine connection, Levi Civita connection and Torsion free connection respectively.*

KEYWORDS*: Smooth manifold, Connection Co-efficient, covariant derivative, affine connection, Torsion free connection, Levi-civita connection, tangent vectors.*

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I. INTRODUCTION

Manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbb{R}^n$. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so that many concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold. Throughout this, all our manifolds are assumed to be smooth, means C^∞, or infinitely differentiable. A curve in three-dimensional Euclidean space is parameterized locally by a single number t as (x (t), y (t), z (t)) while two numbers u and v parameterize a surface as (x (u, v), y(u, v), z(u, v)) where a curve and a surface are considered locally homeomorphism.

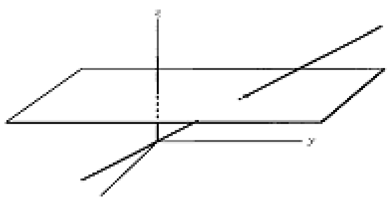


Fig 1.1: The real projective n space $\mathbb{R}^n$ determined by the point on the Line

There are different types of Manifolds given by the following examples on which we will describe the vast descriptions of various deepens that becomes more sophisticated.

II. BASIC DEFINITIONS OF MANIFOLDS:

2.1 Topology: Let X be a non-empty set. A system $\mu = \{\mu_i;\ i \in I, \mu_i \subset X\}$ is called a topology on X, if it contains the following three conditions:

(i) The empty set φ and the set X belong to μ.

(ii) The union of any number of a finite number of sets μ are in the system μ.

(iii) The Intersection of any finite number of sets μ belongs to μ .

The set of μ is called open. The pair (μ, X) consisting of a set X, and a topology μ on X is called a topological space.

2.2 Housdorff Space: A topological space X is called a Housdorff space, if any two distinct Pair's possesses disjoint neighborhoods. Let N_x and N_y possesses disjoint neighborhoods of X then there exists $N_x \cap N_y = \emptyset$.

2.3 Homeomorphism: A function f: X $\rightarrow$ Y between two topological spaces X and Y is called a homeomorphism if it has the following properties:

(i) f is bijective.

(ii) Both f and f^{-1} are continuous.

2.4 C^∞ Diffeomorphism: Let U and V be two open subsets of $\mathbb{R}$. A function f: U $\rightarrow$ V is called C^∞diffeomorphism, if it satisfies the following.

(i) f is homeomorphism.

(ii) If f and f^{-1} are of class C^∞.

2.5 Diffeomorphism: A function f: U $\rightarrow \mathbb{R}^m$ is an isomorphism of C^∞-class and invertible, then f is called a diffeomorphism.

2.6 Local Diffeomorphism

A map f: U $\rightarrow$ V is a local diffeomorphism if and only if f is smooth and det $(\frac{\partial f}{\partial x})/_{x_0} \neq 0$ at each point of $x_0 \in U$, where U and V be open subset in $\mathbb{R}^n$.

2.7 Chart: A chart is a pair $(U, \emptyset)$ consisting of a topological manifold M and an open subset U of M is called the domain of the chart together with a homeomorphism $\emptyset$: U $\rightarrow$ V onto an open set V in $\mathbb{R}^n$.

2.8 Atlas: An atlas of class C^k, on a topological manifold M is a set $\{(U_\alpha, \emptyset_\alpha),\ \alpha \in I\}$ of charts such that, (i) The domains $\{U_\alpha\}$ cover M, that is $\bigcup_{\alpha \in I} U_\alpha$(ii) The homeomorphism $\emptyset_\alpha$ satisfy the following compatibility conditions: the maps $\emptyset_\beta(U_\alpha \cap U_\beta) \rightarrow \emptyset_\alpha(U_\alpha \cap U_\beta)$ open subset in $\mathbb{R}^n$ must be a class of C^k.

III. FUNCTIONS OF MANIFOLDS

3.1 Pull back Function: Let $f : X^n \rightarrow Y^p$ be a smooth map where X^n is an n-dimensional smooth manifold and g is a function on Y^p, a p-dimensional smooth manifold, then the pull back (or reciprocal image) of the function g under the map f is a function on X is

$$f^*(g)(x) = g(f(x)) \qquad where\ x \in X.$$

Example: Let M and N be a smooth manifold and

f : $M \rightarrow N$ is a smooth map

$\uparrow$

$x_0 \mapsto f(x_0) = y_0 \in N$.

Let O_{x_0} be a set of smooth functions at x, O_{y_0} be a set of smooth functions at the image points, then the map f : $M \rightarrow N$ implied by

$$f^* : O_{y_0} \rightarrow O_{x_0}$$

$$\uparrow$$
$$g \mapsto f^*(g) = g \circ f.$$

3.2 Push Forward Function: Let M and N be two smooth manifolds and the map $f: M \to N$ induces a push forward map

$$f_y: T_{x_0} M \leftrightarrow T_{f(x_0)} N \quad \text{for all } x_0 \in M.$$
$$\uparrow$$
$$\overrightarrow{V_{x_0}} \mapsto f_y\,(\overrightarrow{V_{x_0}})$$

defined by

$f_y\,(\overrightarrow{V_{x_0}})g = \overrightarrow{V_{x_0}}\, f * g$, where $\overrightarrow{V_{x_0}}$ be a set of smooth function at x.

3.3 Immersion and embedding: Let f : M→ N be smooth map and let dim M ≤ dim N.

(a) The map f is called an immersion of M into N if f *: $T_p M \to T_{f(p)}$N is an injection (one to one),that is rank f * = dim M.

(b) The map f is called an embedding if f is an injection and an immersion. The image f (M) is called a **sub** manifold of N. [Here, f (M) thus defined is diffeomorphic to M.]

If f is an immersion, f *maps T_pM is isomorphically to an m-dimensional vector subspace of $T_{f(p)}$N since rank f* = dim M, we also find Ker f* = {0}.

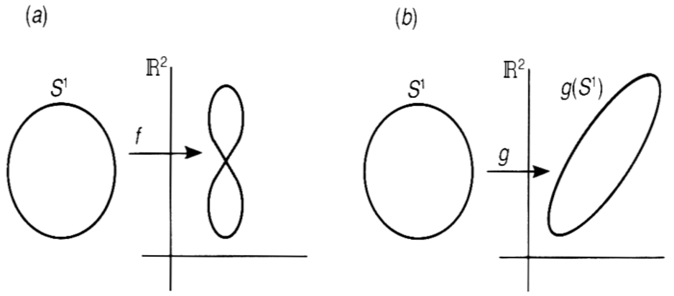


Fig 3.3 (a) An immersion f which is not embedding. (b) An embedding and the submanifold $g(S^1)$.

If f is an embedding, M is diffeomorphic to f (M). Consider a map f: $S^1 \to R^2$from figure 1.8, It is an immersion since a one-dimensional tangent space of S^1 is mapped by f* to a subspace of $T_{f(p)} R^2$. The image f (S^1) is not a sub manifold of R^2 since f is not an injection. Clearly, an embedding is an immersion although the converse is not necessarily true. Now it is clear that; if S^n is embedded by f: $S^n \to R^{n+1}$ then is S^n diffeomorphic to f (S^n).

IV. VARIOUS TYPES OF MANIFOLDS

4.1 Topological Manifold: A topological manifold M of dimension n is a topological space with

(i) M is Housedorff, that is, for each pair P_1, P_2 of M, there exist V_1 , V_2 such that $V_1 \cap V_2 = \emptyset$.

(ii) Each point p ∈M possesses a neighborhood V homeomorphism to an open subset U of $\mathbb{R}^n$.

(iii) M satisfies the second count ability axiom that is, M has a countable basis for its topology.

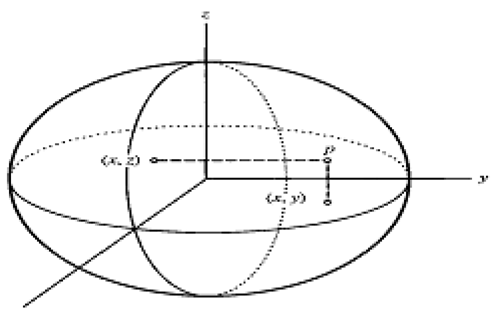


Fig 4.1.1: The surface of a circle is a topological manifold (homeomorphism tos^2).

Examples

1 The sphere S^2 = {(X, Y, Z) ∈ $\mathbb{R}^3$: $x^2 + y^2 + Z^2$= 11} with the subspace topology is a topological manifold.

2 (The circle S^1) The circle S^1 = {(x, y)∈ R^2: $x^2 + y^2$= 1} with the subspace topology is a topological manifold of dimension1.Conditions (i) and (iii) are inherited from the ambient space.

3 (The torus of revolution) the surface of revolution obtained by revolving a circle around an axis that does not intersect it is a topological manifold of dimension 2.

4.2 Riemannian Manifold

A Riemannian Manifold (M,g) is a real differentiable manifold M in which each tangent space is equipped with an inner product$g$, a Riemannian metric which varies smoothly from point to point. It is a pair (M, g) with M a manifold and g a metric g_m on M.

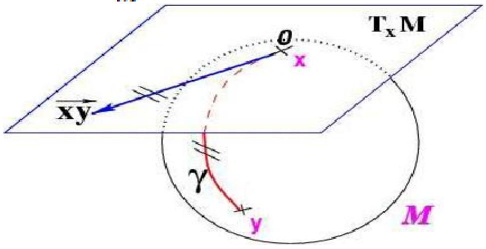


Fig 4.2.1: Riemannian Manifold

Examples

4. Let M be a smooth manifold. A Riemannian metric g on M is a tensor field $g : c_2^\infty(TM) \rightarrow c_0^\infty(TM)$ such that for each p ∈ M the restriction g_p of g to $T_pM \otimes T_pM$ with g_p: (X_p, Y_p) ↔ g(X, Y) (p) is an inner product on the tangent space T_pM. The pair (M, g) is called a Riemannian manifold.

4.3 Pseudo-Riemannian manifold

A pseudo-Riemannian manifold is a differentiable manifold equipped with a non-degenerate, smooth, symmetric metric tensor such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p, q), where both p and q are non-negative.

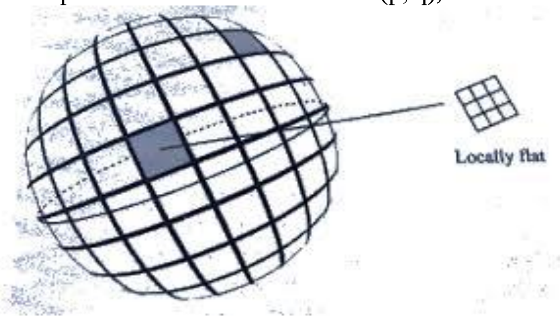


Fig 4.3.1: Pseudo-Riemannian Manifold

Some basic theorems of Riemannian geometry can be generalized to the pseudo-Riemannian case. In particular, the fundamental theorem of Riemannian geometry is true of pseudo-Riemannian manifolds as well. This allows one to speak of the Levi-Civita connection on a pseudo-Riemannian manifold along with the associated curvature tensor. On the other hand, it is not true that every smooth manifold admits a pseudo-Riemannian metric of a given signature; there are certain topological obstructions. Furthermore, a submanifold does not always inherit the structure of a pseudo-Riemannian manifold; for example, the metric tensor becomes zero on any light-like curve. The Clifton–Pohl torus provides an example of a pseudo-Riemannian manifold that is compact but not complete, a combination of properties that the Hopf–Rinow theorem disallows for Riemannian manifolds.

4.4 Differentiable Manifold

If M is an m-dimensional differentiable manifold, it satisfies the following:

(i) M is a topological space;
(ii) M is provided with a family of pairs $\{(U_i, \emptyset_i)\}$;
(iii) $\{U_i\}$ is a family of open sets which covers M, that is $\cup_i U_i = M$, $\emptyset_i$ is a Homeomorphism from U_i onto an open subset U_i' of R^m.

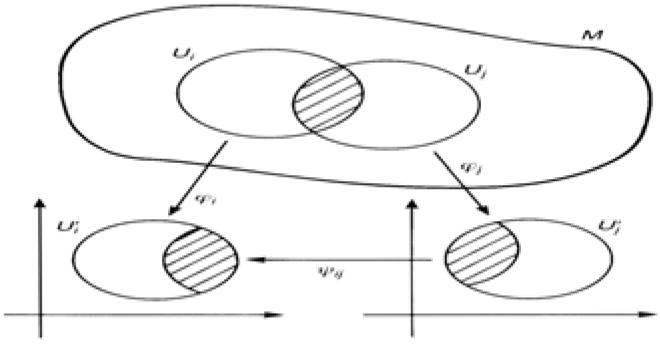


Fig 4.4.1: A homeomorphism $\emptyset_i$ maps U_i onto an open subset $U_i' \in R^m$.

The pair $(U_i, \emptyset_i)$ is called a chart while the whole family $\{(U_i, \emptyset_i)\}$ is called an atlas. The subset U_i is called the coordinate neighborhood while $\emptyset_i$ is the coordinate function or, simply, the coordinate. The homeomorphism $\emptyset_i$ is represented by m functions $\{(X^1(p), X^2(p), \ldots, X^m(p))\}$.From (ii) and (iii), M is locally Euclidean. In each coordinate neighborhood U_i, M looks like an open subset of R^m whose element is $\{(X^1, X^2, \ldots, X^m\}$.

Example

6. We are living on the earth whose surface is S^2, which does not look $(X^1, X^2, \ldots\ldots, x^m)$like R^2 globally. However, it looks like an open subset of R^2 locally.

4.5 Smooth Manifold

A topological manifold M, together with an equivalent class of c^k atlases is called a c^k structure on M and M is called c^k – manifold. If k = ∞ then M is said to be a smooth manifold.

Example

7. Let X^n be an n-dimensional smooth manifold and Y^p be an p-dimensional smooth manifold, then f **:** $X^n \to Y^p$ be a map. The map f is called smooth at a point $x \in X^n$ if $\Psi \circ f \circ \emptyset^{-1}$ is smooth at $\emptyset(x) \in \mathbb{R}^n$ where (U, $\emptyset$) be the coordinate chart at $x \in U$ and (V, $\emptyset$) be the coordinate chart at $y \in V$ .Such that $\emptyset : U \to \mathbb{R}^n$ and $\emptyset^{-1}: \mathbb{R}^n \to U \subset X^n \xrightarrow{f} Y^p \xrightarrow{\Psi} \mathbb{R}^p$ .The mapping is $\Psi \circ f \circ \emptyset^{-1}$: $\mathbb{R}^n \to \mathbb{R}^p$.

V. CONNECTION

A connection which should be thought of as a directional derivative for vector fields. Loosely speaking, this structure by itself is different only for developing analysis on the manifold, while doing geometry requires in addition some way to relate the tangent spaces at different points, i.e. a notion of parallel transport. A connection which should be thought of as a directional derivative for vector fields. The apparatus of vector bundles, principal bundles and connections on them plays an extraordinary important role in the modern differential geometry.

5.1 Definition: A connection on TM is a bilinear map $(TM) \times \Gamma(TM) \to (TM)$ (ξ, X) $\to \nabla_\xi$ X such that $\xi \in M_m$, X, Y $\in \Gamma(TM)$ and $f \in c^\infty(M)$.

Hence the three conditions are as follows:

1. ∇_ξ X $\in M_m$,
2. $\nabla_\xi(fX) = (\xi f) X_m + f(m)\nabla_\xi$ X.
3. ∇_X Y = $\Gamma(TM)$.

which also defined by C^{∞}(M)-linear in X and R-linear in Y and satisfies the product rule

∇_x(f Y) = (Xf) Y + f $\nabla_x Y$ for all $f \in C^{\infty}$(M).

Consider the following two desirable properties for a connection ∇ on (M, g):

1. ∇ is metric: X_g(Y, Z) = g (∇_xY, Z) + g (Y, ∇_xZ).
2. ∇ is torsion free ∇_xY - ∇_y X = [X, Y].

5.2 Connection Co-efficient

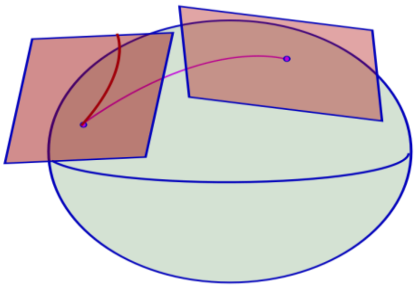
Take a chart (U, ϕ) with the coordinate X = $\phi(p)$ on M, and the functions $\Gamma^{\mu}_{v\lambda}$ is called the connection coefficients by, $\nabla_v e_\mu \equiv \nabla_{eV} e_\mu = e_\lambda \Gamma^{\mu}_{v\lambda}$ Where $\{e_\mu\} = \{\{\partial / \partial x^\mu\}$ is the coordinate basis in T_p M. The connection coefficients specify how the basis vectors change from point to point. Once the action of ∇ on the basis vectors is defined, we can calculate the action of ∇ on any vectors. Let V $= V^\mu e_\mu$ and W $= W^v e_v$ be elements of T_p (M). Then ∇_XW $= V^\mu \nabla_{e\mu}(W^v e_v) = V^\mu (e_\mu[W^\mu] e_v + W^v \nabla_{e\mu} e_v) = V^\mu (\frac{\partial w^\lambda}{\partial x^\mu} + W^v \Gamma^{\lambda}_{\mu v}$ $)e_\lambda$ the connection coefficient is in agreement with the previous heuristic result of covariant derivative.

5.3 Affine connection

Let M be a smooth manifold of dimension n, O_M be the set of smooth function and Γ(TM) be the vector space of vector field. An affine connection on M is a map, denoted by $\nabla(nabla)$ as

: Γ(TM)×Γ(TM)→ Γ(TM),

or, (X, Y) → ∇_XY



such that,

1. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
2. $\nabla_{(X_1+X_2)}Z = \nabla_{X_1}Y + \nabla_{X_2}Y.$
3. $\nabla_{(fX)}Y = f\, \nabla_X Y.$
4. ∇_X(f Y) = X [f] Y + f ∇_X Y

where f ∈ Γ(M) and X, Y, ∈ Γ(TM)

5.4 Torsion Free Connection

Let ∇ be an affine connection on manifold M. Torsion T of the connection ∇ is defined by, T(X, Y) = ∇_X Y - ∇_Y X – [X, Y] For all X, Y ∈ Γ(TM). That is, T (f X, Y) = T (X, f Y) = f T (X, Y).

If T=0, we call ∇ Torsion free connection or a **symmetric connection**.

5.5 The Levi – Civita Connection

We introduce the Levi-Civita connection ∇ of a Riemannian manifold (M, g). This is the most important fact of the general notion of a connection in a smooth vector bundle. Let (M, g) be a Riemannian manifold and let ∇ be an affine connection on M. The covariant derivative of g with respect to ∇ is a multi linear map. We say that ∇ is compatible with the Riemannian metric g.

if ∇_g : Γ(TM)×Γ(TM)×Γ (TM) → O_M

(Z, X, Y) → $\nabla_Z g$ (X, Y) for all X, Y, Z ∈ Γ(TM)

and

$$\nabla_Z\, g\, (X, Y) = Z\, g\, (X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

For all smooth vector fields X, Y and Z on M. The unique torsion-free affine connection on M which preserves the Riemannian metric is known as the Levi - Civita connection on M.

VI. GENERAL THEOREMS AND FUNCTIONS OF MANIFOLDS

Theorem 6.1(Cartan–Hadamard). Suppose M is a complete, connected Riemannian n-manifold with all sectional curvatures less than or equal to zero. Then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.

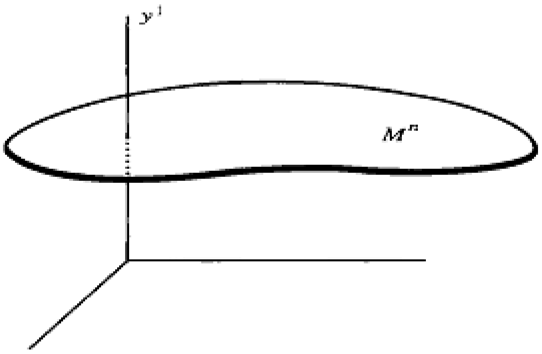


Fig 6.1.1: $Y^1 = f(X^1, \dots, X^n)$ described an n-dimensional manifold of $\mathbb{R}^{n+1}$.

In Figure 6.1.1 we have drawn a portion of the manifold M. This M is the graph of a function $f: \mathbb{R}^n \to \mathbb{R}$ that is, M ={($(X, Y) \in \mathbb{R}^{n+1} \mid y = f(x)$}.
When n = 1, M is a curve; while if
n = 2, then it is a surface.

Theorem 6.2: Every manifold can be given a Riemannian metric.

Proof: If p is a point in a Riemannian manifold (M, g), we define the length or norm of any tangent vector X $\in T_p$ M to be | X |: = $< X, X >^{1/2}$.
If two non- zero vectors X, Y $\in T_p M$ to be unique then there exists a value $\theta \in [0, \pi]$,
satisfying Cos$\theta = < X, Y > (|X||Y|)$.
Here X and Y are orthogonal if their angle is $\pi/2$.
If (M, g) and $(\tilde{M}, \tilde{g})$are Riemannian manifolds, a diffeomorphism $\emptyset$ from M to $\tilde{M}$ is called an isometry if $\emptyset * \ \tilde{g} = g$. An isometric $\emptyset : (M, g) \to (\tilde{M}, \tilde{g})$ is an isometry of M. A composition of isometries and the inverse of an isometry are again isometries, so it is a group.
If ($E_1, E_2, \dots, E_n$) is any local frame for TM and $(\varphi^1, \dots, \varphi^n)$ in its dual coframe a Riemannian metric can be written locally as

$$g = g_{ij} \varphi^i \otimes \varphi^j \quad (1)$$

where $g_{ij} = < E_i, E_j >$ is symmetric in i , j depends on p $\in$ M , in particular in a coordinate frame, g has the form

$$g = g_{ij} dx^i \otimes dx^j \quad (2)$$

By introducing the two terms of the symmetry of equation (1) and (2) on g_{ij} , we get
$g = g_{ij} dx^i dx^j$is a Riemannian metric.
This completes the proof. □

Theorem 6.3: Let (M, g) be a Riemannian manifold then there exists a unique torsion-free affine connection ∇on M compatible with the Riemannian Metric g. This connection is characterized by the identity

$$2g(\nabla_x Y,Z) = X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)$$

For all smooth vector fields X, Y and Z on M.

Proof: Given smooth vector fields X, Y and Z on M, let A (X, Y, Z) be the smooth function on M defined by**,**

$$A(X,Y,Z) = \frac{1}{2}(X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] + g([X,Y],Z)$$

$$- g([X,Z],Y) - g([Y,Z],X)).$$

Then $A(X,Y,Z_1+Z_2) = A(X,Y,Z_1) + A(X,Y,Z_2)$ for all smooth vector fields X, Y, Z_1 and Z_2 on M. Using this identities

$$X,fZ] = f[X,Z] + X[f]Z \quad and \quad [Y,fZ] = f[Y,Z] + Y[f]Z$$

here A (X, Y, f Z) = f A (X, Y, Z) for all smooth real valued functions f and vector fields X, Y and Z on M. On applying Lemma to the transformation Z ↔ A (X, Y, Z), we see that there is a unique vector field ∇_xY on M with the property that A (X, Y, Z) =$g(\nabla_x$Y, Z) for all smooth vector fields X, Y and Z on M.

Moreover $\nabla_{x_1+x_2}$Y = ∇_{x_1}Y +∇_{x_2} Y, $\nabla_x(Y_1 + Y_2)$ = $\nabla_x Y_1 + \nabla_x Y_2$

after calculations, we show that

g (∇_{fx}Y, Z) = A (f X, Y, Z) = f A (X, Y, Z)

= g(f ∇_xY, Z),

g (∇_x (fY), Z) = A (X, f Y, Z) = f A (X, Y, Z) + X[f] g(Y, Z)

= g(f ∇_xY + X[f] Y, Z)

for all smooth real-valued functions for M, so that

$$\nabla_{fx} Y = f \nabla_x Y, \text{ and } \nabla_x(fY) = f\nabla_x Y + X[f]Y$$

these properties show that ∇ is indeed an affine connection on M.

Moreover

A (X, Y, Z) – A (Y, X, Z) = g([X, Y], Z),

so that ∇_xY −∇_yX = [X, Y].

Thus, the affine connection ∇ is torsion-free. Also

$g(\nabla_x y$,Z) + g(Y,∇_xZ) = A (X, Y, Z) + A (X, Z, Y) = X [g (Y, Z)]

showing that, the affine connection ∇ preserves the Riemannian metric. Finally suppose that $\nabla'$is any torsion-free affine connection on M which preserves the Riemannian metric. Then

X [g(Y, Z)] = $g(\nabla'_x$Y, Z) + g(Y, $\nabla'_x Z$),

Y [g(X, Z)] = $g(\nabla'_y$X, Z) + g(X, ∇'_y Z),

Z [g (X, Y)] = $g(\nabla'_Z$X, Y) + g(X, ∇'_ZY).

After calculation (using the fact that ∇' is torsion-free) shows that A (X, Y, Z) = $g(\nabla'_x$Y, Z).

Therefore ∇'_xY = ∇_xY for all smooth vector fields X and Y on M.

This completes the proof of the theorem. □

Theorem 6.4: There is a unique torsion-free metric connection on any Riemannian manifold.

Proof: Assume that g is metric and torsion-free. Then

g (∇_xY, Z) = X_g (Y, Z) - g(Y,∇_xZ)

= X_g(Y, Z) – g (Y, [X, Z]) - g(Y,∇_z X)

$$= g([X,Y],Z) + Y(g(X,Z)) + g(X;\nabla_Y Z),$$

$$0 = -Z(g(X,Y)) + g(\nabla_Z X,Y) + g(X,\nabla_Z Y)$$

$= -Z(g(X,Y)) + g(\nabla_X Z + [Z,X], Y) +$
$g(X, \nabla_y Z + [Y,Z]$

eventually by adding these relations we get

$2g(\nabla_x Y, Z) = X_g(Y,Z) + Y_g(Z,Y) - Z_g(X,Y) -$
$g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$

This formula shows uniqueness and moreover, defines the desired connection. □

Theorem 6.5: For all affine connection ∇ and its curvature $\mathbb{R}^{\nabla}$ is a tenser of rank (1, 3).

Proof: To show $\mathbb{R}^{\nabla}$ is a tenser of rank (1, 3), we have to show that

$\mathbb{R}^{\nabla}(fx,y,z) = \mathbb{R}^{\nabla}(x,fy,z) = \mathbb{R}^{\nabla}(x,y,fz) =$
$f\,\mathbb{R}^{\nabla}(x,y,z)$
$\forall\, x,y,z \in \Gamma(TM)$ And $f \in$ OM

Now by the definition, we get

$\mathbb{R}^{\nabla}(fx,y,z) = \nabla_{fx}\nabla_y z - \nabla_y \nabla_{fx} z - \nabla_{[fx,y]} z$
$= f\,\nabla_x \nabla_y z - \nabla_y(f\,\nabla_x z) - \nabla_{f[x,y]-y(f)x}\, z$

$f\,\nabla_x\nabla_y z - y(f)\,\nabla_x z - f\,\nabla_y\nabla_x z - f\nabla_{[x,y]} z +$
$y(f)\,\nabla_x z$

[Since $\nabla_{fx}(y) = f,\ \nabla_x(fy) = f\,\nabla_x y + x(f)\ \ \nabla_{x_{1+x_2}} = \nabla_{x_1} y + \nabla_{x_2} y$
$= f(\nabla_x\nabla_y z - \nabla_y\nabla_x z - f\nabla_{[x,y]} z) = f\mathbb{R}^{\nabla}(x,y,z)$

Again we have,

Since $\mathbb{R}^{\nabla}(x,y,z) = -\mathbb{R}^{\nabla}(y,x,z)$
$\mathbb{R}^{\nabla}(x,fy,z) = -\mathbb{R}^{\nabla}(fy,x,z)$
$= -f\,\mathbb{R}^{\nabla}(y,x,z)$
$= f\,\mathbb{R}^{\nabla}(x,y,z)$

And finally

$\mathbb{R}^{\nabla}(x,y,fz) = \nabla_x\nabla_y fz - \nabla_y\nabla_x fz - \nabla_{[x,y]} fz$
$= \nabla_x(y(f)z + f\,\nabla_y z) - \nabla_y(x(f)z + f\,\nabla_x z) - [x,y](f)z - f\,\nabla_{[x,y]} z$
$= x(y(f)z) + y(f)\nabla_x z + x(f)\nabla_y z + f\nabla_x\nabla_y z \quad - y(x(f)z) - x(f)\,\nabla_y z \quad - y(f)\nabla_x z$
$- f\,\nabla_x\nabla_y z - [x,y](f)z - f\nabla_{[x,y]} z$

[Since $\nabla_x(fy) = f\nabla_x y + x(f)y$
$\nabla_{fx}(y) = f\ \ \nabla_x y$]

=>

$\mathbb{R}^{\nabla}(x,y,fz) = (xy(f) - yx(f))z + f\,\nabla_x\nabla_y z -$
$\nabla_y\nabla_x z - \nabla_{[x,y]} z) - [x,y](f)z$

$= [x,y](f)z - [x,y](f)z + f(\nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]} z)$
$= f\mathbb{R}^{\nabla}(x,y,z)$

Here $\mathbb{R}^{\nabla}$ is a tenser of rank (1, 3) □

Theorem 6.6: Let (M,g) be a Riemannian manifold. Then the Levi-Civita connection ∇ is a connection on the tangent bundle TM of M.

Proof: It follows from the fact that g is a tensor field that

$g(\nabla_X(\lambda \cdot Y_1 + \mu \cdot Y_2), Z)$

$= \lambda \cdot g(\nabla_X Y_1, Z) + \mu \cdot g(\nabla_x Y_2, Z)$ and

$g(\nabla_{Y_1+Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z)$

for all $\lambda, \mu \in R$ and $X, Y_1, Y_2, Z \in C^\infty(TM)$.

Furthermore, we have for all $f \in C^\infty(M)$

$2\, g(\nabla_X f\, Y, Z) = \{X\,(f \cdot g(Y, Z)) + f \cdot Y\,(g(X, Z)) - Z\,(f \cdot g(X, Y)) + f \cdot g([Z, X], Y) + g([Z, f \cdot Y], X) + g(Z, [X, f \cdot Y])$

$=$
$\{X(f) \cdot g(Y,Z) + f \cdot X(g(Y,Z)) + f \cdot Y\,(g(X,Z))$

$-Z(f) \cdot g(X,Y) - f \cdot Z(g(X,Y)) + f \cdot g([Z,X],Y)$
$+g(Z(f) \cdot Y + f \cdot [Z,Y],X) + g(Z, X(f) \cdot Y + f \cdot [X,Y])\}$
$= 2 \cdot \{X(f) \cdot g(Y,Z) + f \cdot g(\nabla_X Y, Z)\}$
$= 2 \cdot g(X(f) \cdot Y + f \cdot \nabla_X Y, Z)$

And

$2 \cdot g(\nabla_{f \cdot X} Y, Z) = \{f \cdot X\,(g(Y,Z)) + Y\,(f \cdot g(X,Z)) - Z(f \cdot g(X,Y))$
$+g([Z, f \cdot X], Y) + f \cdot g([Z,Y],X) + g(Z, [f \cdot X, Y])\}$
$= \{f \cdot X(g(Y,Z)) + Y\,(f) \cdot g(X,Z) + f \cdot Y\,(g(X,Z))$

$-Z(f) \cdot g(X,Y) - f \cdot Z(g(X,Y)) + g(Z(f) \cdot X, Y) + f.g([Z,X],Y) +$

$f \cdot g([Z,Y],X) + f \cdot g(Z,[X,Y]) - g(Z, Y\,(f) \cdot X$
$= 2 \cdot f \cdot g(\nabla_X Y, Z)$

So, ∇ is a connection on the tangent bundle on TM of M. □

Theorem 6.7: Let (M,g) be a Riemannian manifold. Then the Levi- Civita connection is the unique metric and torsion-free connection on the tangent bundle (TM, M, π).

Proof: The difference $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$ equals twice the skew-symmetric part with respect to the pair (X, Y) of the right-hand side of the equation in the map

$\nabla: C^\infty(TM) \times C^\infty((TM)) \to C^\infty((TM))$

Given by $2\, g(\nabla_x Y, Z) = \{X(g(Y,Z)) + Y\,(g(X,Z)) - Z(g(X,Y)) + g([Z,X],Y) + g([Z,Y],X) + g(Z,[X,Y])\}$.

is called the Levi - Civita connection on M.

This is the same as

$\frac{1}{2}\{g(Z,[X,Y]) - g(Z,[Y,X])\} = g(Z,[X,Y])$

this proves that the Levi-Civita connection is torsion-free.

The sum $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$ equals twice the symmetric part (with respect to the pair (Y, Z) on the right-hand side of ∇ is compatible with the Riemannian metric g.

if $\nabla_g : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to O_M$

$(Z, X, Y) \to \nabla_Z g(X, Y)$ for all $X, Y, Z \in \Gamma(TM)$

and

$$\nabla_Z g(X, Y) = Z\, g(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

. This is exactly

$= \frac{1}{2}\{X(g(Y,Z)) + X(g(Z,Y))\} = X(g(Y,Z)).$

This shows that the Levi-Civita connection is compatible with the Riemannian metric g on M. □

Theorem 6.8: For all affine connection ∇ and metric g, then ∇_g is a tensor.

Proof: We need to show that ∇_g is a tensor, so that

$$\nabla_{fZ}\, g(X,Y) = \nabla_Z\, g(fX,Y)$$
$$= \nabla_Z\, g(X, fY)$$
$$= f\,\nabla_Z\, g(X,Y) \quad \text{for all X, Y, Z } \in \Gamma\text{(TM)}.$$

(a) $\nabla_{fZ}\, g(X,Y) = fZg(X,Y) - g(\nabla_{fZ}X, Y) - g(X, \nabla_{fZ}Y)$

$= fZg(X,Y) - g(f\nabla_Z X, Y) - g(X, f\nabla_Z Y)$

$= fZg(X,Y) - fg(\nabla_Z X, Y) - fg(X, \nabla_Z Y)$

$= f[Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)]$

$= f\,\nabla_Z\, g(X,Y)$

(b) $\nabla_Z\, g(fX,Y) = Zg(fX,Y) - g(\nabla_Z(fX), Y) - g(fX, \nabla_Z Y)$

$= Zg(fX,Y) - g(f\nabla_Z X, Y) - g(X, f\nabla_Z Y)$

$= Zfg(X,Y) - g(Z(f)X + f\nabla_Z X, Y) - fg(X, \nabla_Z Y)$

$= Z(f).g(X,Y) + f Z. g(X,Y) - Z(f)\, g(X,Y) - f g(\nabla_Z X, Y) - f g(X, \nabla_Z Y)$

$= f[Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)]$

$= f\,\nabla_Z\, g(X,Y)$

(c) $\nabla_Z\, g(X, fY) = \nabla_Z\, g(fY, X)$

[Due to symmetry of g]

$= f\nabla_Z\, g(Y,X)$ [by (b)]

$= f\nabla_Z\, g(X,Y).$

Hence, ∇_g is a tensor. □

Theorem 6.9: ∇ is a torsion free connection if and only if $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$ for all ν, λ.

Proof: If ∇ is a torsion free connection then for any v, λ we get

$T(e_\nu, e_\lambda) = \nabla_{e_\nu} e_\lambda - \nabla_{e_\lambda} e_\nu - [e_\nu, e_\lambda]$

$= \Gamma^{\mu}_{\nu\lambda} e_\mu - \Gamma^{\mu}_{\lambda\nu} e_\mu - 0$

$$= (\Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\nu})\, e_\mu$$
$$= 0$$

Conversely, let $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$ then

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$= X^\nu Y^\lambda (\nabla_{e_\nu} e_\lambda - \nabla_{e_\lambda} e_\nu - [e_\nu, e_\lambda]$

$= X^\nu Y^\lambda (\Gamma^{\mu}_{\nu\lambda} e_\mu - \Gamma^{\mu}_{\lambda\nu} e_\mu - 0)$

$= X^\nu Y^\lambda (\Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\nu})\, e_\mu$

$= 0$ (since $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$)

Since, $e_\mu \neq 0$ so that $\Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\nu} = 0$ or $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$.

Since T is a torsion tensor and $T(e_\nu, e_\lambda) = 0$ for all e_ν, e_λ.

So, ∇ is a torsion free Connection.

This completes the proof □

Theorem 6.10: For all ∇ its torsion tensor T^∇ is a tensor of rank (1, 2).

Proof: We need to show that $T^\nabla(fX, Y) = T^\nabla(X, fY) = f\,T^\nabla(X, Y)$ for all $X, Y \in \Gamma(TM)$.

We know $T^{\nabla}(X, Y) = \nabla_X Y - \nabla_y X - [X, Y]$

$T^{\nabla}(fX, Y) = \nabla_{fx} Y - \nabla_y fX - [fX, Y]$

$= f\nabla_X Y - (Y(f)X + f\nabla_y X) - [fX, Y]$

$= f\nabla_X Y - Y(f)X - f\nabla_y X - [fX, Y]$

Take any $g \in 0_M$, then

$[fX, Y]\, g = fX\,Y(g) - Y(fX(g))$

$= fX\,Y(g) - Y(f).\,X(g) - fY\,X(g)$

$= f(X\,Y(g) - Y\,X(g)) - Y(f).X(g)$

$= f[X, Y](g) - Y(f).X(g)$

so $[fX, Y] = f[X, Y] - Y(f).X$

$T^{\nabla}(fX, Y) = f\nabla_X Y - Y(f)X - f\nabla_y X - [fX, Y] + y(f)X$

$= f(\nabla_X Y - \nabla_y X - [X, Y]) = fT^{\nabla}(X, Y)$

Again, we know

$T^{\nabla}(X, Y) = -T^{\nabla}(Y, X)$

$T^{\nabla}(X, fY) = -T^{\nabla}(fY, X)$

$= -fT^{\nabla}(Y, X)$

$= -(-)fT^{\nabla}(Y, X)$

$= fT^{\nabla}(Y, X)$

So, T^{∇} is a tensor of rank (1, 2) □

Theorem 6.11: Let d_{im} M = n. Then the dimension of tangent vector on manifold T_{xo} M is also n.

Proof: Let (U, φ) be a coordinate chart at x_o in manifold M. Then any $f \in Ox_o$ be a smooth function at x_o and it can be represented by smooth function of n-variables.

$$f\ o\ \varphi^{-1} = \bar{f}\ (x^1, x^2, \ldots\ldots\ldots x^n) \in \mathbb{R}^n$$

Now by using Taylor's series at x_o

$$\bar{f}\ (x^1, x^2, \ldots\ldots\ldots x^n) = \bar{f}\ (x^1, x^2, \ldots\ldots\ldots x^n) \neq \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x^i}\Big|x^i = x_o{}^i (x^i - x_o{}^i)$$

$$+ \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{f}}{\partial x^i\, \partial x^j}\Big|x^i = x_o{}^i\ (x^i - x_o{}^i)(x^j - x_o{}^j) + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots$$

Then for any tangent vector $\overrightarrow{V_{x_o}}\,\bar{f} = \overrightarrow{V_{x_o}}\,\bar{f}\,(x^1, x^2, \ldots\ldots\ldots x^n) + \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x^i}\Big|x^i = x_o{}^i\,\overrightarrow{V_{x_o}}\ (x^i - x_o{}^i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \bar{f}}{\partial x^i\, \partial x^j}\Big|x^i = x_o{}^i\,\overrightarrow{V_{x_o}}\ (x^i - x_o{}^i)(x^j - x_o{}^j) + \ldots\ldots\ldots\ldots$

$= 0$ (derivative of any scalar) $+ \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x^i}\Big|x^i = x_o{}^i\,\overrightarrow{V_{x_o}}\ (x^i) + v + v + \ldots$

$\overrightarrow{V_{x_o}} \bar{f} = \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x^i} \overrightarrow{V_{x_o}} (x^i) \; |x^i = x_o{}^i$

denote $v^i = \overrightarrow{V_{x_o}} (x^i) \in \mathbb{R}$, they are called coordinates of $\overrightarrow{V_{x_o}}$ in the chart (u, φ).

thus $\overrightarrow{V_{x_o}}$ and $\forall f$, $\overrightarrow{V_{x_o}} f$ is represented in (u, φ) as a linear combination.

$\overrightarrow{V_{x_o}} \bar{f} = \sum_{i=1}^{n} v^i \frac{\partial \bar{f}}{\partial x^i} \; |x^i = x_o{}^i$

$\overrightarrow{V_{x_o}} = \sum_{i=1}^{n} v^i \frac{\partial \bar{f}}{\partial x^i} \; |x^i = x_o{}^i$

Conclusion:

Any tangent vector $\overrightarrow{V_{x_o}}$ can be written as a linear combination of the partial derivatives in (i).

In particular, $\frac{\partial}{\partial x^i}$ are the tangent vectors, we need to show that the set of vectors $\{\frac{\partial}{\partial x^i}, i = i, n$ are linearly independent.

We assume that $c_1 \frac{\partial}{\partial x^1} + c_2 \frac{\partial}{\partial x^2} + \ldots + c_n \frac{\partial}{\partial x^n} = 0$

For some scalars $c_1, c_2, \ldots\ldots\ldots\ldots c_n \in \mathbb{R}$

Let us apply the L.H.S to a function $f = x^1, f = x^2, \ldots, f = x^n$

$$c_1 \frac{\partial}{\partial x^1} (x^1) + c_2 \frac{\partial}{\partial x^2} (x^1) + \ldots\ldots\ldots\ldots + c_n \frac{\partial}{\partial x^n} (x^n)$$

$c_1 \; .1 + 0 + 0 + \cdots \ldots \ldots .+0 = 0$

$c_1 \; = 0$

Similarly for $f = x^2$

$$c_1 \frac{\partial}{\partial x^1} (x^2) + c_2 \frac{\partial}{\partial x^2} (x^2) + \ldots\ldots\ldots\ldots + c_n \frac{\partial}{\partial x^n} (x^n)$$

$0 + c_2 \; .1 + 0 + \cdots \ldots \ldots .+0 = 0$

$c_2 \; = 0$ and $c_n \; = 0$ for $t = x^n$

Thus, any tangent vector is a linear combination of $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$, ….,$\frac{\partial}{\partial x^n}$ and the tangent vectors $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$, ……..,$\frac{\partial}{\partial x^n}$ are linearly independent and that form a bans and contains exactly n vectors.

Therefore d_{im} (Tx_o, M) = n. □

VII. CONCLUSIONS:

Connections of manifolds are of central importance in modern geometry in large part because they allow a comparison between the local geometry at one point to another point. It is a well-known fact that, a Riemannian metric on a differentiable manifold induces a Riemannian metric on its submanifold and hence, a Riemannian connection on the manifold induces a Riemannian connection on its submanifold. From this paper we have come to know that an affine connection is typically given in the form of a covariant derivative, which gives a means for taking directional derivatives of vector fields: the infinitesimal transport of a vector field in a given direction. A general theorem of manifolds with connections is a great importance in the basic field of modern applied geometry. Generally the notion of a connection makes precise the idea of transporting data along a curve or family of curves in a parallel and consistent manner.

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