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On the computation of transfer matrices for age processes in discrete-time $SM[K]/PH[K]/1/FCFS$ queue

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Abstract： *In this paper, we study a discrete-time queuing system: multiple types of customers arrive in batches, the arrival process is a semi-Markov process, first-come-first-served, and each customer's service time obeys its own distribution. The age process of this discrete-time queue* $SM[K]/PH[K]/1/FCFS$ *is analyzed in detail, and some additional variables are introduced to construct a Markov chain about the age process, so that the transfer matrix of the age process can be calculated.*

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I. Introductory

Modern communication networks need to handle all types of data, which vary in terms of capacity, etc. Modern supply chains are also designed to meet the different needs of customers, which is equivalent to a queuing system in which customers are of different types. In life, we also often encounter the phenomenon of batch arrival, for example, in the manufacturing system, the need to process the parts of the batch to the processing plant; and then in the inventory system, customer demand in accordance with the composite Poisson process to arrive at the system. All these phenomena can be described as batch-arrival queuing systems. Due to the need for design and functional analysis of these stochastic systems, we will study a class of discrete-time queuing models with multiple types of customers that arrive in batches.

Let's study discrete-time queue $SM[K]/PH[K]/1/FCFS$. The system has k customer types (k is a positive integer), all of which join a queue and are served on a “first-come-first-served” basis. The customer arrival process is a discrete semi-Markov process, and the service time of each customer follows a discrete distribution PH, independent of each other and independent of their arrival process. There are many studies of queuing-like systems in the literature, but there are still very few studies of $SM[K]/PH[K]/1/FCFS$.

II. Customer arrival process

The customer arrival process is a discrete semi-Markov process. Customers are divided into k types and arrive in batches (let us call them customer batches). In order to characterize the nature of customer batches, at first define a set of integer strings:

$$\aleph = \{J_k : J_k = j_1 j_2 \cdots j_{n_k}, 1 \le j_i \le K, 1 \le i \le n_k, 1 \le k \le N\}$$

where N is the total number of distinct integer strings in the set $\aleph$ and n_k is the total number of customers in the kth batch. We assume N is finite. A string $J = j_1 j_2 \cdots j_n \in \aleph$ represents n customers of type . We denote this customer batch by J. Thus, there are N different batch representations.

Consider a Semi-Markov chain $\{(\xi_n, \tau_n), n \ge 0\}$ with m_α phases , where ξ_n is the phase of the Semi-Markov chain at the instant of arrival of the nst customer batch, and τ_n is the time between the arrival of the (n-1)th batch and the arrival of the nth batch (i.e., the transfer time interval). The arrival of the customer batch is associated with the transfer of the semi-Markov process in the following way. Let J_n be the string representation of the customer lot associated with the nth transfer, i.e.: a customer batch J_n arrives at the moment of transfer. Definition:

$$P\{\xi_n = j, \tau_n = t, J_n = J \mid \xi_{n-1} = i\} = p_{J,i,j}(t), 1 \le i, j \le m_\alpha, n \ge 1, J \in \aleph,$$

Where t is a positive integer, and the variable $p_{J,i,j}(t)$ is the conditional probability that a customer batch J arrives after the time t since the arrival of the previous batch and the phase of the semi-Markov process is from i changed to j. Let $D_{\alpha,J}(t)$ be a $m_\alpha \times m_\alpha$ matrix with (i,j) element $p_{J,i,j}(t)$. Thus, the sequence of matrices $\{D_{\alpha,J}(t), t \ge 1, J \in \aleph\}$ provides all the information about the labeled transfer semi-Markov arrival process.

Definition:

$$D_\alpha(t) = \sum_{J\in\aleph} D_{\alpha,J}(t), t \ge 1, \quad D_{\alpha,J} = \sum_{t=1}^{\infty} D_{\alpha,J}(t), J \in \aleph, \quad D_\alpha = \sum_{J\in\aleph} D_{\alpha,J} = \sum_{t=1}^{\infty} D_\alpha(t).$$

Thus, the matrix D_α is the transfer matrix of the embedded Markov chain of the semi-Markov chain $\{(\xi_n, \tau_n), n \ge 0\}$ at the moment of transfer.

III. Customer queuing process

Upon arrival of a customer batch, all customers join the queue according to their order in the batch. All batches are served on a "first-come-first-served" basis by a single service counter. For each batch, customers are served in the order in which they are in the batch. Let $q(t)$ be a string of customer types in the queue at t time, which can be obtained from the customers who are likely to be served at $t-1$ time and the customers who have arrived. If $q(t) = j_1 j_2 \cdots j_n$, then there are n customers in the system at the time t, the j_1 type of customer is being served, and the j_2 type of customer is the first customer in the queue, $\cdots$, and the j_n type of customer is the last customer in the queue. These n customers will be served in the same order $j_1 j_2 \cdots j_n$. If the next customer of type J arrives in a batch, the queue will change to $q(t)+J$. If the next customer finishes the service, then the queue will become $j_2 j_3 \cdots j_n$, and j_2 start to receive the service. Of course, it is different for the case that a certain type of customer has the priority of service in his/her customer batch.

IV. Service time

The service times of each customer follow a discrete PH distribution, which are independent of each other and of their arrival processes. For a customer of type k, its service time s_k obeys a PH distribution whose matrix is given by $\{m_k, \alpha_k, T_k\}$, where m_k is the number of bits of the PH distribution, α_k is an initial probability vector, and T_k is a transfer matrix. We assume that the service time of any customer is at least 1. This is because for a discrete PH distribution, the transfer time of a bit phase is 1, and $\alpha_k e = 1\ (1 \le k \le K)$, so the customer cannot be served as soon as he receives the service. Remember $T_k^0 = (I - T_k)e$, where I is the unit matrix. The service time of a customer batch is the sum of the service times of all customers in the batch. Since the structure of the PH distribution is closed under convolution, the service time s_J of a customer batch of type J also has a discrete-time PH distribution. The matrix is denoted by $\{m_k, \alpha_k, T_k\}$, where for $J = j_1 j_2 \cdots j_n$, there is

$$m_J = \sum_{i=1}^{n} m_{j_i}, \alpha_J = (\alpha_{j_1}, 0, \cdots, 0)$$

$$T_J = \begin{pmatrix} T_{j_1} & T_{j_1}^0\alpha_{j_2} & & & \\ & T_{j_2} & T_{j_2}^0\alpha_{j_3} & & \\ & & \ddots & \ddots & \\ & & & T_{j_{n-1}} & T_{j_{n-1}}^0\alpha_{j_n} \\ & & & & T_{j_n} \end{pmatrix}, T_J^0 = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T_{j_n}^0 \end{pmatrix}$$

V. Analysis of the generalized age process

We construct a Markov chain for the age process $a_g(t)$ by introducing some additional variables. The introduction of additional variables is discussed as follows:

(1) At time $a_g(t) < 0$, we introduce three additional variables $I_a(t), J(t), I_s(t)$. Since $a_g(t) < 0$,the system is empty at this point, the next customer batch will take another moment $-a_g(t)$ to arrive. We use the Markov chain $\{\xi_n, n \geq 0\}$ to define $I_a(t)$, $I_a(t) = \xi_n$,if the nst customer lot is the one to be served, then $I_a(t)$ represents the state of the semi-Markov chain when the upcoming customer lot arrives at the system, and $J(t), I_s(t)$ represent the type of the customer lot and its initial service phase, respectively.

(2) At time $a_g(t) \geq 0$, we still introduce three additional variables: $I_a(t), J(t), I_s(t)$. Since there are customers in the system at this time, $I_a(t) = \xi_n$.If the nst customer batch is the customer batch being served, then $I_a(t)$ represents the state of the semi-Markov chain when the customer batch being served arrives at the system, and then $J(t), I_s(t)$ represent the type of the customer batch being served and its service phase at the time, respectively. As we can see, the system is empty at the time $a_g(t) < 0$, and the customer batch will take $-a_g(t)$ moments to arrive, so the state of the semi-Markov chain at the time of its arrival, as well as its type and initial service phase, should be unknown. However, we can advance this time so that the change of $I_a(t), J(t), I_s(t)$ in advance does not affect the transfer of the state of the customer batch after it arrives at the system. Thus, our construction is feasible, and we find that the established process $\{a_g(t), I_a(t), J(t), I_s(t), t \geq 0\}$ is Markovian during the time that a customer batch is served: this is because the service time of the batch is determined by a latent Markov chain (the PH distribution), which $I_a(t), J(t)$ do not change during this period, and the value of $a_g(t)$ is increased by 1. As the system completes the service of a customer batch, the value of $I_a(t)$ changes according to the Markov chain and determines the arrival intervals, also $a_g(t)$ changes. $J(t)$ is also determined by the Markov chain and $I_s(t)$ by the initial distribution of service times. Therefore, the process $\{a_g(t), I_a(t), J(t), I_s(t), t \geq 0\}$ is also Markovian at the moment when a customer batch is served. Summarizing the above, we can see that the process about $a_g(t)$ we have built is a Markov process.

After analysis, the transfer probability matrix of this Markov chain $\{a_g(t), I_a(t), J(t), I_s(t), t \geq 0\}$ constructed for the age process is:

$$P_g = \begin{array}{c} \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \overset{\begin{array}{cccccc} \cdots & -2 & -1 & 0 & 1 & 2 \;\cdots \end{array}}{\begin{pmatrix} \ddots & \ddots & & & & & \\ \cdots & 0 & B & & & & \\ \cdots & 0 & 0 & B & & & \\ \cdots & A_3 & A_2 & A_1 & A_0 & & \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & \\ \cdots & A_5 & A_4 & A_3 & A_2 & A_1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}} \leftarrow \text{level } 0, \text{ i.e. } a_g(t)=0 \qquad (5.1)$$

VI. Solving the transfer probability matrix for the age process

In the probability transfer matrix of Eq. (5.1) above, the probability $P\{a_g(t+1)=y|a_g(t)=x\}$ is greater than zero only when $y \le x+1$; since an increase in time by one does not increase age by more than one, the probability is zero when $y > x+1$. Based on the transfer from one age to another, and the range of values of the additional variables for each age Markov chain, it can be obtained that each matrix block element of the matrix P_g is a small $(m_a \cdot m_{tot}) \times (m_a \cdot m_{tot})$ matrix.

(1) When $a_g(t)=x, x \le -1$,the system is empty and the next batch of customers will not arrive for $-x$ moment, at $t+1$ moment, the batch either has not yet arrived, or has just arrived and started to be served, but at least it is certain that the batch has not left the system after being served. Then $a_g(t)$ should be Increase by 1 with time, and both $I_a(t)$ and $I_a(t+1)$ refer to the state of the semi-Markov chain when the customer batch arrives at the system, both $J(t)$ and $J(t+1)$ represent the type of the customer batch, and both $I_s(t)$ and $I_s(t+1)$ represent its initial service phase. Thus, when $a_g(t)=x, x \le -1$,

$$P\{a_g(t+1)=y, I_a(t+1)=j', J(t+1)=J', I_s(t+1)=i' \\ |a_g(t)=x, I_a(t)=j, J(t)=J, I_s(t)=i\} \\ = \begin{cases} 1, & \text{if } y=x+1, j=j', J=J', i=i' \\ 0, & \text{else} \end{cases}$$

By the dictionary ordering of the states and the fact that the probability is 1 only when $y=x+1, j=j', J=J', i=i'$.Thus $B=I_{(m_a \cdot m_{tot})\times(m_a \cdot m_{tot})}$.The probability of moving from $a_g(t)=x$ to any other age is zero.

(2) When $a_g(t)=x, x \ge 0$,there is at least one customer batch receiving service in the system, the customer batch may continue to receive service at the time $t+1$, or may leave the system after service.

If the customer batch receiving service at the time t continues to receive service at the time $t+1$, then the transfer matrix block is A_0,then there should be: $a_g(t+1)=x+1$,and both $I_a(t)$ and $I_a(t+1)$ refer to the state of the semi-Markov chain when the customer batch arrives at the system, and both $J(t)$ and $J(t+1)$ represent the type of the customer batch, but $I_s(t)$ represent the service phase at the time t, $I_s(t+1)$ represent the service phase at the time $t+1$, and the change of the two is determined by the transfer matrix of the service phase. Thus,

$$P\{a_g(t+1)=x+1, I_a(t+1)=j', J(t+1)=J', I_s(t+1)=i' \\ |a_g(t)=x, I_a(t)=j, J(t)=J, I_s(t)=i\} \\ = \begin{cases} (T_J)_{i,i'}, & \text{if } j=j', J=J' \\ 0, & \text{else} \end{cases} \qquad (6.1)$$

In order to find A_0, we write this matrix block $(A_0)_{(m_a \cdot m_{tot})\times(m_a \cdot m_{tot})}$ as $m_a \cdot m_a$ matrix sub-blocks of $m_{tot} \times m_{tot}$, based on a dictionary ordering of the states:

$$A_0 = \begin{pmatrix} A_0^{1,1} & A_0^{1,2} & \cdots & A_0^{1,m_a} \\ A_0^{2,1} & A_0^{2,2} & \cdots & A_0^{2,m_a} \\ \vdots & \vdots & \cdots & \vdots \\ A_0^{m_a,1} & A_0^{m_a,2} & \cdots & A_0^{m_a,m_a} \end{pmatrix}$$

where $A_0^{i,j}(1 \le i, j \le m_a)$ is a $m_{tot} \times m_{tot}$ subblock of the matrix, analyzing each $A_0^{i,j}$, by (6.1) and the definition of T_{tot},has:

$$A_0^{i,j} = \begin{cases} T_{tot}, & i = j, 1 \le i, j \le m_a \\ 0, & else \end{cases}$$

By the definition of the Kronecker product we have $A_0 = I \otimes T_{tot}$.

If the customer batch that receives the service at the time t leaves the system after the service is finished at the time $t+1$, then the transfer matrix block is $A_s, s \ge 1$, then $a_g(t+1)$ is related to the next customer batch and should be $a_g(t+1) = a_g(t) + 1 - \tau_{n(t+1)+1}$. And $I_a(t+1)$ is the state of the semi-Markov chain when the next customer batch arrives at the system, $J(t+1)$ and $I_s(t+1)$, denote its type and initial service phase, so that

$$\begin{aligned} &P\{a_g(t+1) = x+1-s, I_a(t+1) = j', J(t+1) = J', I_s(t+1) = i' \\ &\quad \left| a_g(t) = x, I_a(t) = j, J(t) = J, I_s(t) = i \right\} \\ &= P\{\tau_{n(t+1)+1} = s, \xi_{n(t+1)+1} = j', J_{n(t+1)+1} = J' \left| \xi_{n(t+1)} = j \right\} \\ &\quad \cdot P\{\text{A customer batch of type J in service phase i to complete the service}\} \\ &\quad \cdot P\{\text{The type of the customer batch is J}',\text{and its service phase is i}'\} \\ &= (D_{a,J'}(s))_{j,j'} (T_J^0)_i (\alpha_{J'})_{i'}, s \ge 1 \end{aligned} \tag{6.2}$$

In order to find A_s, we write this matrix block $(A_s)_{(m_a \cdot m_{tot})\times(m_a \cdot m_{tot})}$ as $m_a \cdot m_a$ matrix sub-blocks of $m_{tot} \times m_{tot}$, based on a dictionary ordering of the states:

$$A_s = \begin{pmatrix} A_s^{1,1} & A_s^{1,2} & \cdots & A_s^{1,m_a} \\ A_s^{2,1} & A_s^{2,2} & \cdots & A_s^{2,m_a} \\ \vdots & \vdots & \cdots & \vdots \\ A_s^{m_a,1} & A_s^{m_a,2} & \cdots & A_s^{m_a,m_a} \end{pmatrix}$$

where $A_s^{i,j}(1 \le i, j \le m_a)$ is the matrix subblock of $m_{tot} \times m_{tot}$:

$$A_s^{i,j} = (\sum_{k=1}^{N} D_{a,J_k}(s))_{i,j} \otimes (T_{tot}^0 \alpha(J_k)), 1 \le i, j \le m_a$$

By the definition of Kronecker product $A_s = \sum_{i=1}^{N} D_{a,J_i}(s) \otimes (T_{tot}^0 \alpha(J_i)), s \ge 1$.

Combining the above yields the transfer matrix P_g.

Theorem 1 The transfer matrix P_g of the Markov chain $SM[K]/PH[K]/1/FCFS$ with respect to age is a random matrix.

Prove:

$$\sum_{s=0}^{\infty} A_s \cdot e_{m_a \cdot m_{tot}} = A_0 \cdot e_{m_a \cdot m_{tot}} + \sum_{s=1}^{\infty} A_s \cdot e_{m_a \cdot m_{tot}}$$

$$= (I \otimes T_{tot}) \cdot (e_{m_a} \otimes e_{m_{tot}}) + \sum_{s=1}^{\infty} (\sum_{i=1}^{N} D_{a,J_i}(s) \otimes (T_{tot}^0 \alpha(J_i)) \cdot (e_{m_a} \otimes e_{m_{tot}}))$$

$$= (I \cdot e_{m_a}) \otimes (T_{tot} \cdot e_{m_{tot}}) + \sum_{s=1}^{\infty} \sum_{i=1}^{N} (D_{a,J_i}(s) \cdot e_{m_a}) \otimes (T_{tot}^0 \alpha(J_i) e_{m_{tot}})$$

$$= e_{m_a} \otimes (T_{tot} \cdot e_{m_{tot}}) + \sum_{s=1}^{\infty} \sum_{i=1}^{N} (D_{a,J_i}(s) \cdot e_{m_a}) \otimes T_{tot}^0$$

$$= e_{m_a} \otimes (T_{tot} \cdot e_{m_{tot}}) + D_a \cdot e_{m_a} \otimes T_{tot}^0$$

$$= e_{m_a} \otimes (T_{tot} \cdot e_{m_{tot}}) + e_{m_a} \otimes T_{tot}^0$$

$$= e_{m_a} \otimes (T_{tot} \cdot e_{m_{tot}} + T_{tot}^0)$$

$$= e_{m_a} \otimes e_{m_{tot}} = e_{m_a \cdot m_{tot}}$$

Thus, the transfer matrix is a random matrix.

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