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A Variational Homotopy Perturbation Method Approach to the Nonlinear Equations Governing MHD Jeffery-Hamel Flow in the Presence of Magnetic Field

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ABSTRACT*: The MHD Jeffrey-Hamel flow incorporating magnetic field is theoretically investigated using the hybrid variational Homotopy perturbation method (VHPM). The strongly nonlinear partial differential equations governing the flow in polar coordinates are first converted to an ordinary differential equation using the Cauchy-Riemann equation or Stream function formulation. Thereafter, the resulting nonlinear PDEs in the governing parameters are then solved for their effects on the geometry of the flow. The Reynold number, Hartmann number and angle of inclination have profound influence on the velocity profile of the flow as shown in the tables and graphs. The solution obtained confirmed the method is accurate, efficient and agree with those in literature.*

KEYWORDS*: Jefferey-Hamel Flow, Magnetic Field, Variational Iteration Method (VIM), Homotopy Perturbation method (HPM), Variational Homotopy Perturbation Method (VHPM)*

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I. INTRODUCTION

Jefferey-Hamel flow proposed by Jeffery [1] and Hamel [2] in 1915 and 1916 is the flow of steady incompressible fluid through a convergent/divergent channel between non-parallel inclined walls from a source or sink. The governing equations from this phenomenon are nonlinear, hence analytical solutions are difficult to come by. However, using simplifying assumptions, they obtained an exact solution via similarity transform for a 2-D incompressible where the radial acceleration is in the z-direction.

This flow has many useful applications in fluid mechanics, civil, environmental, mechanical, and biomechanical engineering. Due to these myriads of applications in the industry and academia, several scholars have given considerable attention and extended the original study to include different parameters for varying values of the Hartmann number and angle of inclination. In recent times, numerical and semi-analytical techniques have been preferred to solve several problems for approximate solution. The solutions obtained are convergent with high degree of accuracy.

This problem has been extensively studied and well documented in literature [3-6]. Khidir [7] employed spectral-Homotopy perturbation method to the governing equations of Jeffery-Hamel flow. The result obtained showed agreement when compared with other semi-analytical methods like ADM HPM, HAM both for the convergent and divergent channels. Al-Jawary [8] investigated the problem using three distinct semi-analytical iterative schemes namely: Temimi-Ansari (TAM), Daftardar-Jafari (DJM) and Banach contraction method (BCM). The result showed convergence and computationally elegant. [9-10] have carried out an analytical investigation using differential transform method (DTM) incorporating the magnetic parameter and nanoparticles. The Adomian decomposition method (ADM) have been used to examine the Jeffery-Hamel flow for analytical solution by Ganji [11]. Sheikholeslami [12] studied the Jeffery-Hamel flow in the presence of high magnetic field and nanoparticles employing the Adomian decomposition method. The comparison of the error between numerical and Adomian decomposition methods for different values of the Hartmann parameter and angle showed concurrence. [13-14] utilised the Homotopy Analysis method (HAM) to consider the Jeffery-Hamel problem with suction or injection for non-parallel walls. [15-16] analysed the problem using the Homotopy perturbation method (HPM) and Hermite-Pade approximation to seek analytical solution. Motsa et al [17] used Spectral-Homotopy Analysis method (SHAM) to solve the nonlinear equations governing the Jeffery-

Hamel problem. The Optimal Homotopy Asymptotic method and Wavelet techniques have been used for the analysis for the MHD nonlinear equations governing the Jeffery-Hamel flow by [18-19]. Akinpelu et al [20] investigated the same problem using Galerkin weighted residual method. Their study showed, there is close agreement between all the methods used. [21] employed the Neutral network optimized techniques to investigate the nonlinear equations of Jeffery-Hamel flow for analytical solution.

In this present article, we study the same problem in the presence of magnetic field using the coupling of variational iteration method (VIM) and Homotopy perturbation method (HPM). This to the best of our knowledge hasn't been applied so is novelty. The organization of the study is as follows: Section one takes an in-depth look at previous literatures and methods from different academics who have tackle this problem. The mathematical formulation of the fundamentals of the Jeffery-Hamel flow is presented in section two. Section 3 & 4 gives the basics of the variation iteration method (VIM) and He's Homotopy perturbation method. The coupling of VIM and HPM is presented in section 5. The mathematical procedure to the problem via VHPM is contained in section 6 and section 7 takes the results and discussion in tables and graphically.

II. MATHEMATICAL FORMULATION OF JEFFERY-HAMEL FLOW

We consider a convergent/divergent rigid wall which makes an angle of 2α, where a steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink. The walls are divergent if $\alpha > 0$ and convergent if $\alpha < 0$. Furthermore, assuming the velocity is to be purely radial, and the flow parameter remains unchanged along the $2-$direction. The flow depends on r and θ, where r and θ are radial and angular coordinates so that the velocity, $v = (u(r,\theta), 0)$ as shown in Figure 1.

Following Schlichting [5], the continuity, Navier-Stokes and Maxwell's equations in polar coordinates are given in reduced form as

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(ru(r,\theta)\right) = 0 \tag{1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \upsilon\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right] - \frac{\sigma B_0^2 u(r,\theta)}{\rho r^2} \tag{2}$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{2\upsilon}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0 \tag{3}$$

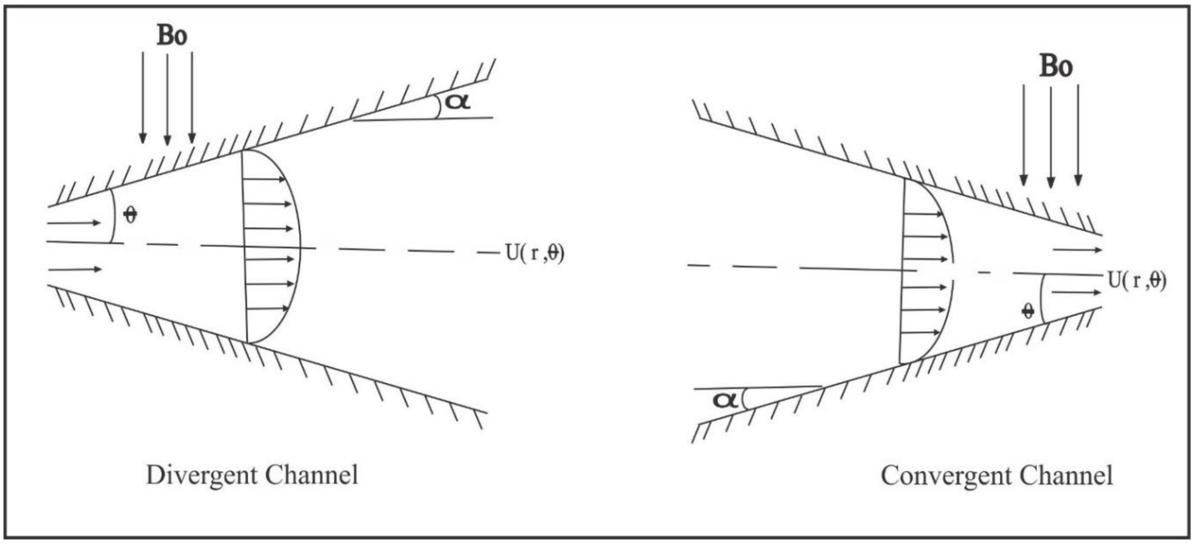

Figure 1. Schematic configuration for $2D$ Jeffery-Hamel flow with Magnetic Field

Where $B_0, P, \rho, \sigma, \upsilon$ denotes electromagnetic induction, pressure of the fluid, density of the fluid, conductivity of the fluid and coefficient of kinematic viscosity of the fluid.

For purely radial flow, $\frac{\partial u}{\partial \theta} = 0$ we define the flow parameter in the form

$$f(\theta) = ru(r,\theta)$$

$$u(r,\theta) = \frac{f(\theta)}{r} \tag{4}$$

Introducing dimensionless parameters as

$$F(\eta) = \frac{f(\theta)}{f_{max}}, \eta = \frac{\theta}{\alpha} \tag{5}$$

Plugging Eq. (5) into Eqs. (2) & (3), and eliminate the pressure term, we obtain an ordinary differential equation for the normalized function profile, $F(\eta)$

$$\frac{\partial^3 F(\eta)}{\partial \eta^3} + 2\alpha Re F(\eta)\frac{\partial F(\eta)}{\partial \eta} + (4 - Ha)\alpha^2 \frac{\partial F(\eta)}{\partial \eta} = 0 \quad (6)$$

Subject to the boundary conditions

$$F(0) = 1, F'(0) = 0, F(1) = 0 \quad (7)$$

Where Re is the Reynold's number defined by

$$Re = \frac{f_{max}\alpha}{\upsilon} = \frac{U_{max} r\alpha}{\upsilon}\begin{pmatrix} divergent\ channel: \alpha > 0, U_{max} > 0 \\ convergent\ channel: \alpha < 0, U_{max} < 0 \end{pmatrix}$$

Where U_{max} is the velocity at the centre of the channel ($r = 0$) and $Ha = \sqrt{\frac{\sigma B_0^2}{\rho \upsilon}}$ is the Hartmann number.

III. HE'S VARIATIONAL ITERATION METHOD (VIM)

The basic idea of the VIM is as follows

Consider the ordinary differential equation of the form

$$Ly + N(y) = f(x), \qquad x \in I \quad (8)$$

Where L and N are linear and nonlinear operators respectively, and $f(x)$ is any given inhomogeneous terms defined for $x \in I$

We defined a correctional functional for Eq. (8) as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)\left(Ly_n(\tau) + N(\tilde{y}_n(\tau)) - f(\tau)\right)d\tau \quad (9)$$

Where $\lambda(\tau)$ is a Lagrange multiplier obtained through variational theory, $y_n(x)$ is the nth approximation of $y(x)$ and $\tilde{y}_n(x)$ is a restricted variation meaning $\delta\tilde{y}_n(x) = 0$

By imposing the variation of both sides of Eq. (9) and taking the restricted variation we obtained

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta(\int_0^x \lambda(\tau) Ly_n(\tau)d\tau \quad (10)$$

$$\delta y_{n+1}(x) = \delta y_n(x) + \left[\lambda(\tau)(\int_0^\tau Ly_n(\xi)d\xi\right]_{\tau=0}^{\tau=x} - \int_0^x \lambda^1(\tau)(\int_0^\tau L\delta y_n(\xi))d\xi \quad (11)$$

Now by applying the stationary condition, the value of the Lagrange multiplier, $\lambda(\tau)$ can be found. Then the successive approximations, $y_n(x), n = 0,1,2,3$ .... Can be found in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)\left(Ly_n(\tau) + N(y_n(\tau)) - f(\tau)\right)d\tau \quad (12)$$

The exact solution is then obtained as the limit of the successive approximations from Eq. (12)

$$y(x) = \lim_{n\to\infty} y_n(x)$$

IV. HOMOTOPY PERTURBATION METHOD (HPM)

In this section, the fundamentals of the Homotopy perturbation method as proposed by He. J. Huan is discussed

Consider a functional differential equation of the form

$$\mathcal{A}(u) - f(r) = 0, r \in \Omega \quad (13)$$

Subject to the boundary condition

$$\mathcal{B}\left(u, \frac{\partial u}{\partial t}\right) = 0, r \in \mathcal{T} \quad (14)$$

where

$\mathcal{A}$ is a differential operator

$\mathcal{B}$ is a boundary operator

$u(x,t)$ is an unknown function

$\mathcal{T}$ is the boundary of the domain Ω

$f(x,t)$ is a known analytic function

Decomposing the operator, $\mathcal{A}$ into two parts comprising linear, ($\mathcal{L}$) and nonlinear ($\mathcal{N}$)

$$\mathcal{A} = \mathcal{L} + \mathcal{N} \quad (15)$$

In view of Eq. (3), we rewrite Eq. (1) in the form

$$\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0 \quad (16)$$

Embedding an artificial parameter p on Eq. 16) as follows

$$\mathcal{L}(u) + p(\mathcal{N}(u) - f(r)) = 0 \quad (17)$$

where $p \in [0,1]$ is the embedding or artificial parameter.

Next, we construct a Homotopy, $\mathcal{H}(r,p): \Omega \times [0,1] \longrightarrow \Re$ to Eq. (17) that satisfies

$$\mathcal{H}(r,p) = (1 - p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{L}(v) + \mathcal{N}(v) - f(r)] = 0 \quad (18)$$

and

$$\mathcal{H}(r,p) = \mathcal{L}(v) - \mathcal{L}(u_0) + p\mathcal{L}(u_0) + p[\mathcal{N}(v) - f(r)] = 0 \quad (19)$$

Where $u_0(x)$ is the initial approximation which satisfies the boundary condition.

Putting $p = 0$ and $p = 1$ into Eq. (19), we obtain the following equations

$$\left.\begin{aligned}\mathcal{H}(r,0) &= \mathcal{L}(v) - \mathcal{L}(u_0)\\ \mathcal{H}(r,1) &= \mathcal{A}(u) - f(r)\end{aligned}\right\} \quad (20)$$

Clearly as p changes monotonically from zero to unity, $\mathcal{H}(r,p)$ changes from $u_0(x)$ to $u(x)$. This is called deformation, whereas the terms $\mathcal{L}(v) - \mathcal{L}(u_0)$ and $\mathcal{A}(u) - f(r)$ are homotopic to each other.

Now we consider a power series solution in p as follows

$$v = \sum_{n=0}^{\infty} p^{(n)} v_n \quad (21)$$

The approximate solution of Eq. (21) can be obtained by setting $p = 1$

$$u(x) = \lim_{p\to 1} v_n = v_0 + v_1 + v_2 + \cdots \quad (22)$$

Similarly, the nonlinear term, $\mathcal{N}(u)$ can be expressed in He's polynomial [27]

$$\mathcal{N}(u) = \sum_{n=0}^{\infty} p^{(m)} H_m(v_0 + v_1 + \cdots + v_m) \quad (23)$$

Where $H_m(v_0 + v_1 + \cdots + v_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[\mathcal{N}\left(\sum_{k=0}^{m} p^k v_k\right)\right]_{p=0}, m = 0,1,2,\ldots$ (24)

where,

$$H_0 = \mathcal{N}(u_0)$$

$$H_1 = u_1\mathcal{N}'(u_0)$$

$$H_2 = u_2\mathcal{N}'(u_0) + \frac{1}{2}\mathcal{N}_1^2\mathcal{N}''(u_0)$$

$$H_3 = u_3\mathcal{N}'(u_0) + u_1u_2\mathcal{N}''(u_0) + \frac{1}{6}\mathcal{N}_1^3\mathcal{N}'''(u_0)$$

$$H_4 = u_4\mathcal{N}'(u_0) + (\frac{1}{2}u_2^2 + u_1u_3)\mathcal{N}''(u_0) + \frac{1}{2}u_1^2u_2\mathcal{N}_1^3\mathcal{N}'''(u_0) + \frac{1}{24}u_4^3\mathcal{N}^{(iv)}(u_0)$$

V. VARIATIONAL HOMOTOPY PERTURBATION METHOD (VHPM)

Consider a functional equation of the form

$$\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0$$

According to VIM, we construct a correctional functional of Eq. (16) as follows.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)[\mathcal{L}u_n(\xi) + Nu_n(\xi) - f(\xi)]d\xi \quad (25)$$

Where λ is the Lagrange multiplier, $\mathcal{L}$ is the integral or differential operator, N is the nonlinear operator and $f(r)$ is an analytic function

Now, applying Homotopy we obtain

$$\sum_{n=0}^{\infty} p^{(n)}u_n = u_0(x) + p\int_0^x \lambda(\xi)\left[\sum_{n=0}^{\infty} p^{(n)}\mathcal{L}\big(u_n(\xi)\big) + \sum_{n=0}^{\infty} p^{(n)}\mathcal{N}\big(\tilde{u}_n(\xi)\big)\right]d\xi - \int_0^x \lambda(\xi)f(\xi)\,d\xi \quad (26)$$

VI. ANALYTICAL PROCEDURE VIA VHPM

In this section, we apply VHPM to the ordinary differential equation in Eq. (6) subject to (7). We procced as follows

Firstly, we rewrite Eq. (6) in the form

$$\frac{\partial^3 F(\eta)}{\partial\eta^3} + \lambda F(\eta)\frac{dF(\eta)}{d\eta} + \gamma\frac{dF(\eta)}{d\eta} = 0 \quad (27)$$

Where $\lambda = 2\alpha Re$

$\gamma = (4 - Ha)\alpha^2$

The correction functional of Eq. (26) gives the form

$$F_{n+1}(\eta) = F_n(\eta) + \int_0^\eta \lambda(\tau)\left[\frac{\partial^3 F(\tau)}{\partial\eta^3} + \gamma\frac{dF(\tau)}{d\eta} + \lambda F(\eta)\frac{dF(\tau)}{d\eta}\right] = 0 \quad (28)$$

$$F_{n+1}(\eta) = F(0) + \eta F'(0) + \frac{\eta^2}{2}F''(0) + \int_0^\eta \lambda(\tau)\left[\frac{\partial^3 F(\tau)}{\partial\eta^3} + \gamma\frac{dF(\tau)}{d\eta} + \lambda\tilde{F}(\eta)\frac{d\tilde{F}(\tau)}{d\eta}\right] = 0$$

(28)

Using the condition in Eq. (7) and the Lagrange multiplier, we obtain

$$F_{n+1}(\eta) = 1 + \frac{\eta^2}{2}\sigma + \int_0^\eta \frac{(\tau-\eta)^2}{2}\left[\frac{\partial^3 F(\tau)}{\partial\eta^3} + \gamma\frac{dF(\tau)}{d\eta} + \lambda\tilde{F}(\eta)\frac{d\tilde{F}(\tau)}{d\eta}\right] = 0 \quad (29)$$

Applying Homotopy perturbation method to Eq. (29), we have

$$\sum_{n=0}^{\infty} p^{(n)} u_n = F(0) + p\int_0^\eta \frac{(\tau-\eta)^2}{2}\left[\frac{\partial^3}{\partial\eta^3}(u_0 + pu_1 + p^2u_2 + \cdots) + \gamma\frac{d}{d\eta}(u_0 + pu_1 + p^2u_2 + \cdots) + \lambda\sum_{n=0}^{\infty} p^{(n)}H_n\right] = 0 \quad (30)$$

Where $H_n{}^{'s}$ are the He's polynomials defined in Eq. (24)

Equating coefficients of the powers of p on both sides of Eq. (30), we have the following expressions

$$p^{(0)}: u_0 = 1 + \frac{\sigma}{2}\eta^2$$

$$p^{(1)}:\ u_1 = \int_0^{\eta} \frac{(\tau-\eta)^2}{2}\left(\frac{\partial^3 u_0}{\partial \eta^3} + \gamma \frac{du_0}{d\eta} + \lambda H_0\right)$$
$$p^{(2)}:\ u_2 = \int_0^{\eta} \frac{(\tau-\eta)^2}{2}\left(\frac{\partial^3 u_1}{\partial \eta^3} + \gamma \frac{du_1}{d\eta} + \lambda H_1\right)$$
$$p^{(3)}:\ u_3 = \int_0^{\eta} \frac{(\tau-\eta)^2}{2}\left(\frac{\partial^3 u_2}{\partial \eta^3} + \gamma \frac{du_2}{d\eta} + \lambda H_2\right)$$
$$\vdots$$
$$p^{(n)}:\ u_n = \int_0^{\eta} \frac{(\tau-\eta)^{n-1}}{n-1}\left(\frac{\partial^3 u_{n-1}}{\partial \eta^3} + \gamma \frac{du_{n-1}}{d\eta} + \lambda H_{n-1}\right) \quad (31)$$

Where

$$H_0 = f_0 f_0^{'}$$
$$H_1 = f_0 f_1^{'} + f_1 f_0^{'}$$
$$H_2 = f_0 f_3^{'} + f_1 f_2^{'} + f_2 f_0^{'}$$
$$H_3 = f_0 f_4^{'} + f_1 f_3^{'} + f_2 f_2^{'} + f_3 f_1^{'} + f_4 f_0^{'}$$
$$H_4 = f_0 f_4^{'} + f_1 f_3^{'} + f_2 f_2^{'} + f_3 f_1^{'} + f_4 f_0^{'} \quad (32)$$

The approximate solution of the problem is given by the expression

$$F(\eta) = \lim_{p \to 1} u_n \quad (33)$$

Solving Eq. (31) give the iterative solutions for the first, second and third approximates as

$$F_0(\eta) = 1 + \frac{\sigma}{2}\eta^2$$
$$F_1(\eta) = -\frac{2\,\alpha Re\sigma}{4!}\eta^4 - \frac{(4-Ha)\alpha^2\sigma}{4!}\eta^4 - \frac{2\alpha Re\sigma^2}{240}\eta^6$$
$$F_2(\eta) = \left(\frac{\alpha^2 Re^2\sigma}{360} + \frac{\sigma\alpha^3 Re}{45} + \frac{1}{45}\sigma\alpha^2\right)\eta^6 + \left(\frac{\alpha^3 Re^2\sigma^2}{280} + \frac{(\alpha Re\sigma)^2}{560}\right)\eta^8 + \left(\frac{\sigma^3 Re}{2700} - \frac{\sigma^2(2\alpha Re)^3}{129600} - \frac{(\sigma\alpha^2 Re)^2}{32400} - \frac{\sigma^2\alpha^5 Re}{3240}\right)\eta^{10} - \left(\frac{(\alpha Re\sigma)^3}{95040} + \frac{(\alpha Re)^2\sigma^3}{47520}\right)\eta^{12} - \frac{\sigma^4(\alpha Re)^3}{2620800}\eta^{14} \quad (34)$$

The three-term approximate solution for $F(\eta)$ is given by

$$F(\eta) = F_0(\eta) + F_1(\eta) + F_2(\eta) + \cdots$$
$$F(\eta) = 1 + \frac{\sigma}{2}\eta^2 - \frac{2\,\alpha Re\sigma}{4!}\eta^4 - \frac{(4-Ha)\alpha^2\sigma}{4!}\eta^4 - \frac{2\alpha Re\sigma^2}{240}\eta^6 \quad (35)$$

Imposing the given boundary condition, $F(1) = 0$, and setting $\alpha = 5, Re = 500, Ha = 0$ in Eq. (35), we obtain the value of the constant, $\sigma = -4.56.80072132$

Now substituting the value of σ into Eq. (35), we obtain the concentration profile of the problem.

VII. RESULTS AND DISCUSSION

In this subsection, we analyse the concentration profile under the influence of angle of contact, Reynold number and Hartmann number**.** In the section hereafter, we present our result in tables and graphs.

Table 1. **Comparison between Numerical results and Variational Homotopy Perturbation Method**

η	VHPM Solution	Numerical Solution	$Error = \lvert F_{NM} - F_{VHPM}\rvert$
0	1.0000	1.000000	1.00000
0.1	1.187830	1.18812	0.00029
0.2	4.713290	5.70419	0.99090
0.3	20.4623	20.5120	0.04970
0.4	63.2448	63.3004	0.05560
0.5	153.793	152.8791	0.91390
0.6	318.7690	318.7720	0.00030
0.7	590.7510	590.7620	0.01100
0.8	1008.240	1008.50	0.26000
0.9	615.680	616.800	1.12000
1.0	2463.410	2463.52	0.11000

Table 2: Comparison of Constant values, σ for different values of α, Ha and Re

$Ha = 50$			$Ha = 100$	$Ha = 150$
Re	α	σ	σ	σ
	5	0.03166	o.047244	0.09302

100	7	0.022514	0.03352	0.065574
	10	0.015707	0.23346	0.045455
200	3	0.022857	0.026667	0.03200
	6	0.0113636	0.013245	0.015873
	9	0.0075614	0.0088106	0.0105541

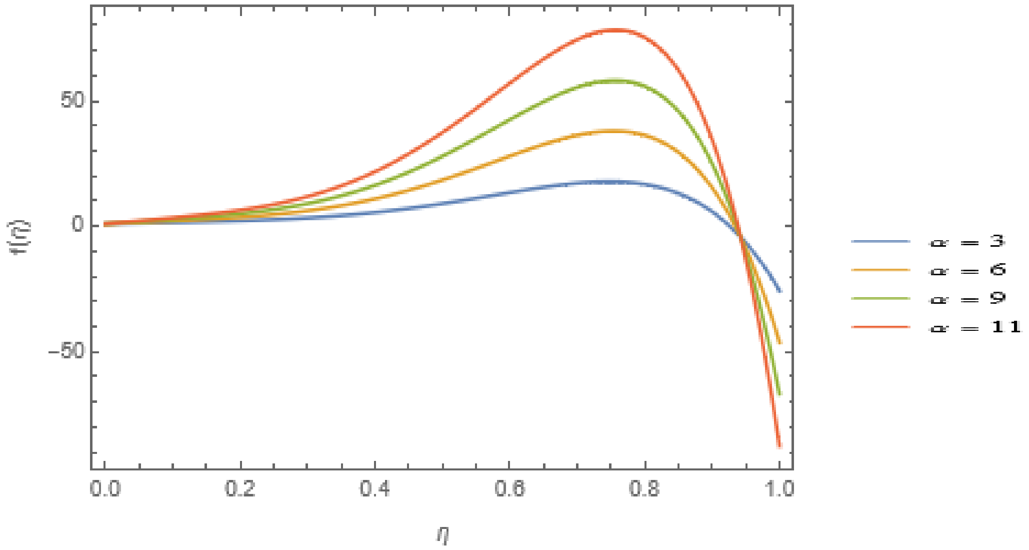

Figure 1. Velocity profile for constant values of Ha, Re and various values of α for diverging channel

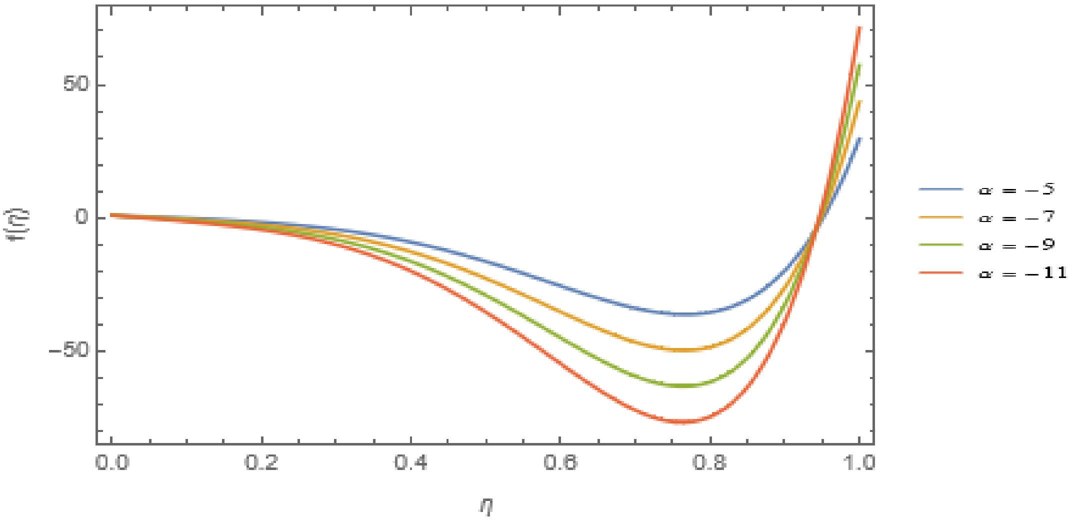

Figure 2. Velocity profile for constant values of Ha, Re and various values of α for converging channel.

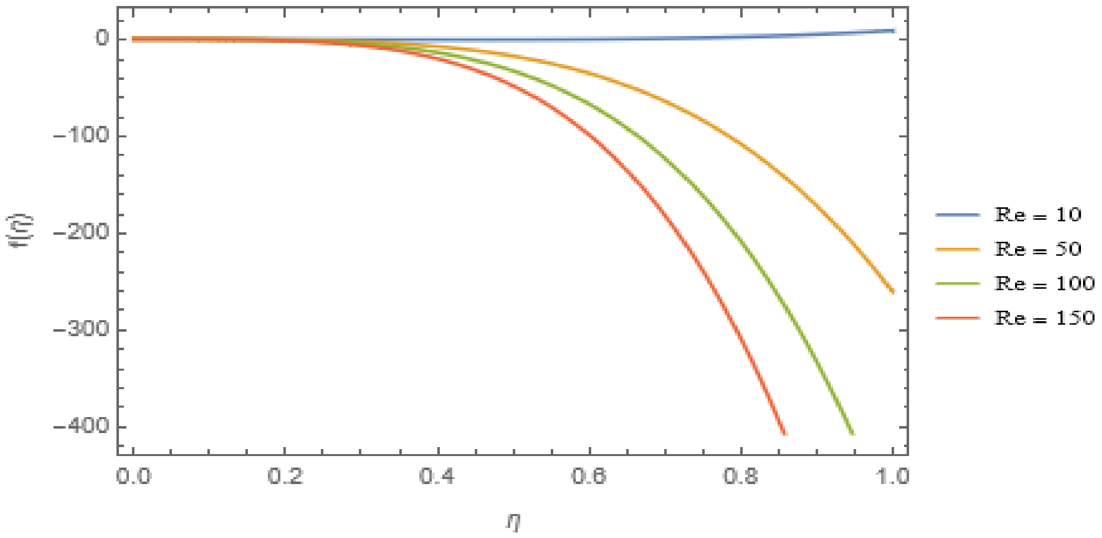

Figure 3. Velocity profile for constant values of Ha, α and various values of Re for converging channel

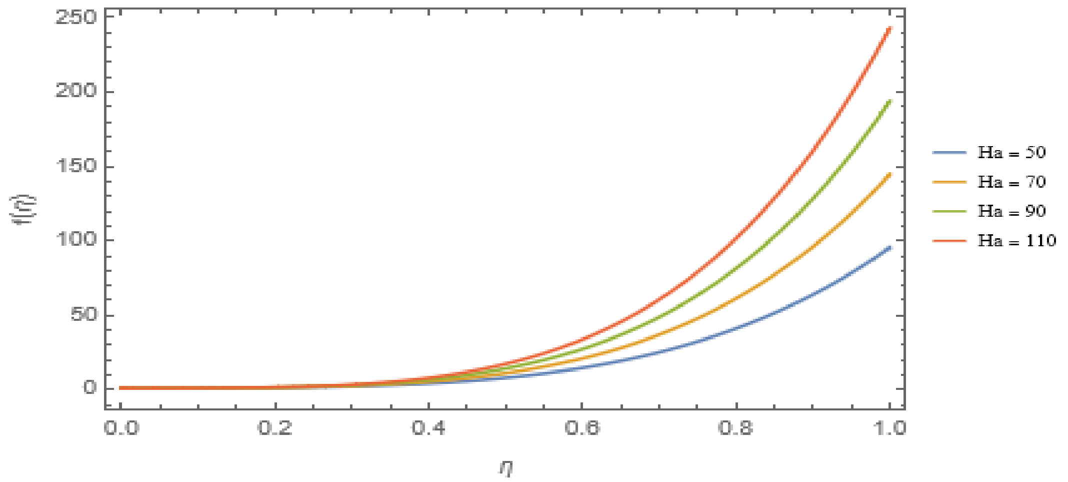

Figure 4. Velocity profile for constant values of Ha and Re for different values of Ha for converging channel.

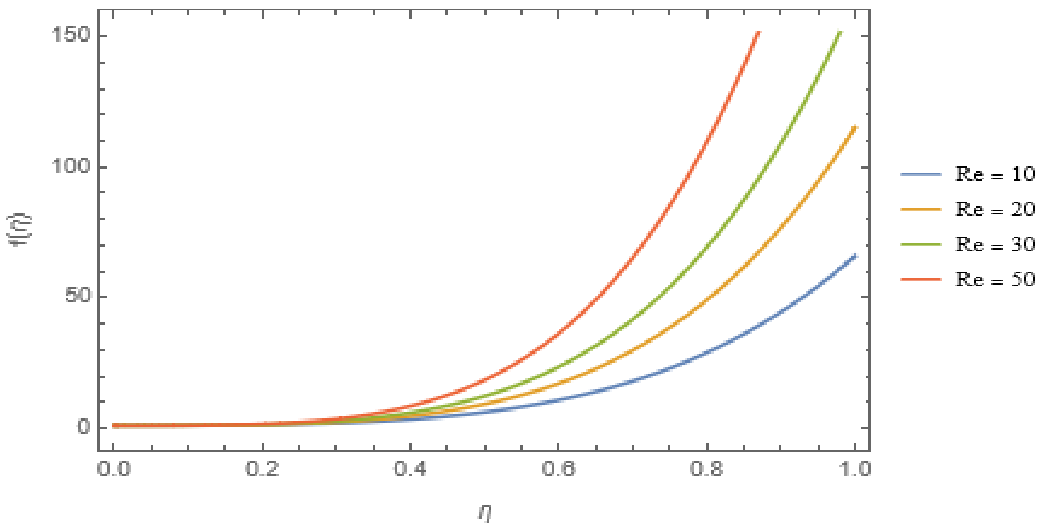

Figure 5. Velocity profile for constant values of Ha and α and various values of Re for converging channel.

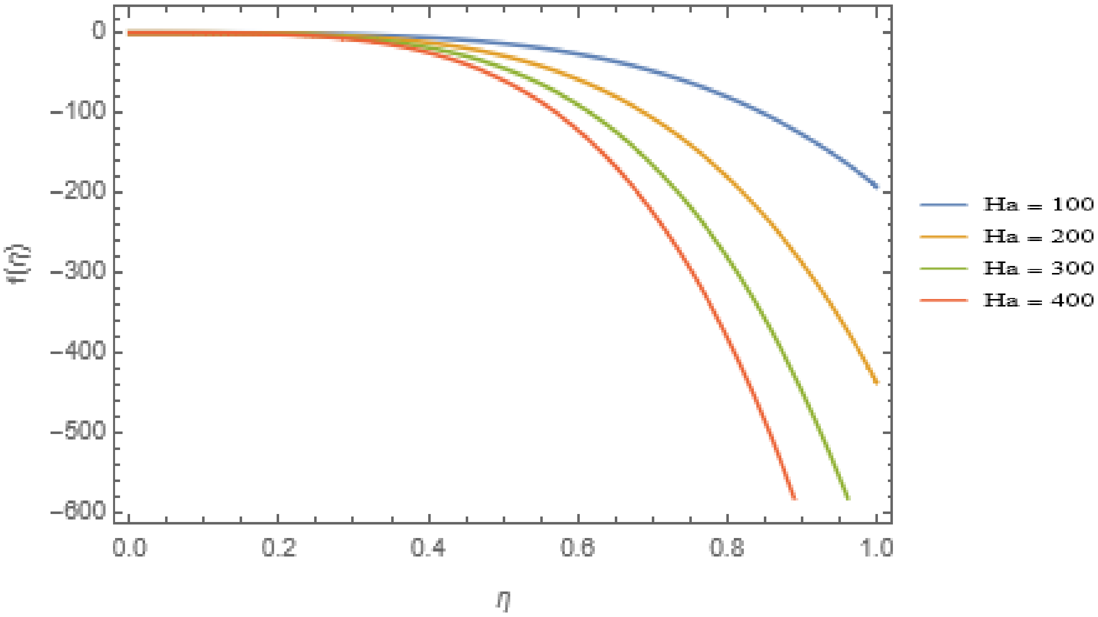

Figure 6. Velocity profile for constant values of Ha and α and various values of Re for converging channel

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