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The mathematics behind the Rayleigh waves
by

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ABSTRACT
Rayleigh waves are acoustic type of waves that are produced naturally through seismic events or artificially in labs. They are used in engineering to determine the shear wave velocity which helps predict soil and rock properties to sustain large structures.
Consider a harmonic wave travelling along the x-axis with both longitudinal and shear components. If the x_1, x_3 represents the ground while x_2 represents the depth into the earth. The displacements u_1, u_2, and u_3 are modeled as a wave propagating with a constant speed and with an amplitude that decreases exponentially from the ground surface. The ground surface is free and so the stress vector on the ground surface is 0.
KEY WORDS: *Rayleigh waves equations, shear wave velocity, poison ratio.*

I. INTRODUCTION

The idea of using Rayleigh wave to determine the shear wave velocity in soils and the rock layers is to reduce the cost and get the work done quickly. This technique is cost-effective and very reliable. However, this method works very well when the ground is under natural condition.

In 1885, Lord Rayleigh investigated earthquakes and modeled the study as a linearized problem of elasticity. In this paper, we will solve the Rayleigh wave equation in function of the Poisson ratio of the medium and study the properties of the roots in order to understand their behavior near the critical Poisson ratio. Section 2 is about the mathematics behind the Rayleigh waves equation.

II. RAYLEIGH WAVES

We recall the main ingredients used in geo-technical: stress and strain tensors, Lame parameters, Young modulus and Poisson ratio. Denote by $\mathbb{R}^3$the Euclidean 3-dimensional space. A point in $\mathbb{R}^3$will be denoted by $x = (x_1, x_2, x_3)$.

STRESS AND STRAIN

Hooke's law for a spring: Force=(Stiffness)(Elongation) extends to most material and can be formulated as STRESS=E(STRAIN), where E is the Young modulus of elasticity. Stress measures average force per unit area (with units Pa), strain is a geometric measure of deformation, and the Young modulus characterizes the stiffness of the material.

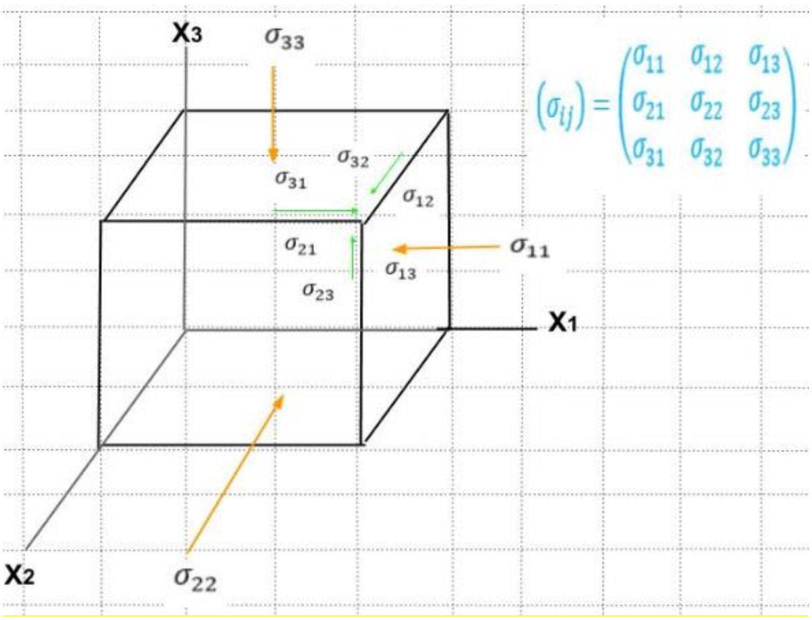
Stress tensor

Stress is defined as the tension or pressure exerted on a material object.
Stress tensor is the stress (force per unit area) at a point in D. The stress tensor can be written as

$$\sigma = (\sigma_{ij})$$

The first index i specifies the direction in which the stress component acts, and the second index j identifies the orientation of the surface upon which it is acting. σ_{ij} is the component in the j-th direction to a surface unit normal in the i-direction.

Figure 1: Stress tensor



<u>Strain tensor</u>

Strain is defined as the change in shape or size of a body due to deforming force applied on it. Strain tensor measures how much a given deformation differ from a rigid motion. It is a dimensionless quantity. If $y = F(x)$ is a deformation. This can also be expressed as

$$\varepsilon = (\varepsilon_{ij}).$$

The Cartesian components of the strain tensor are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \text{ i=1,2,3, j=1,2,3}$$

u here represents the symmetric matrix.

Stress and strain are derived from the Hooke's law. Using this law, we will have:

$$\sigma = 2\mu\varepsilon + \lambda T_r(\varepsilon)I \qquad (1)$$

Where T_r is the trace of ε and I is the identity. This equation can also be expressed as:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\left(\sum_{k=1}^{3} \varepsilon_{kk}\right)\delta_{ij}.$$

λ and μ are known as lame parameters .These properties of the soil (λ *and* μ) are measure directly from the uniaxial test.

(λ, μ) Can be expressed in terms of the young modulus E and Poisson ratio ν.

<u>Young's modulus (E) </u>relates stress and strain.

Itis the measure of the stiffness of an elastic material. It is defined as the ratio of stress to strain. This can be written as E= $\frac{\sigma}{\varepsilon}$

E here is the constant of proportionality between stress and strain. This is one of the mechanical Properties of the soil.

<u>Poisson ratio (ν):</u> Describes expansion or contraction of material in direction perpendicular to loading. It is also one of the properties of the soil.

$$\nu = \frac{\text{transverse expansion}}{\text{axial compression}}$$

ν is dimensionless and usually; $0 \le \nu \le 0.5$. *The* Relation between (λ, μ) *and* (E, ν).

For a string-like object laid out along the x-direction of the coordinate system, Hooke's law for isotropic and homogenous materials leads to proportionality between the tensions in x-direction,

$$\sigma_{xx} = p \qquad (2)$$

And the strains it provokes in directions parallel and orthogonal to it.

$$U_{xx} = \frac{p}{\in}, \qquad U_{xx} = U_{zz} = -V\frac{p}{\in} \qquad (3)$$

In an isotropic material, there are no internal directions defined which can be used to construct such a relation and this means that only tensors at our disposal are strain tensor, σ_{ij}, itself and the kronecker delta, δ_{ij}, multiplied with the trace $\sum_k U_{kk}$ which is the only scalar quantity that can be formed from a linear combination of strain tensor components.
The most general linear tensor relation between stress and train in an isotropic material therefore becomes,

$$\sigma_{ij} = 2\mu U_{ij} + \lambda\delta_{ij} \sum_k U_{kk}$$

Explicitly, we find for the diagonal elements of the stress tensor;

$$\sigma_{xx} = (2\mu + \lambda)U_{xx} + \lambda(U_{yy} + U_{zz})$$
$$\sigma_{yy} = (2\mu + \lambda)U_{yy} + \lambda(U_{zz} + U_{xx})$$
$$\sigma_{zz} = (2\mu + \lambda)U_{zz} + \lambda(U_{xx} + U_{yy})(4)$$

And for the off-diagonal elements,

$$\sigma_{xy} = \sigma_{yx} = 2\mu U_{xy}$$
$$\sigma_{yz} = \sigma_{zy} = 2\mu U_{yz}$$
$$\sigma_{zx} = \sigma_{xz} = 2\mu U_{zx}$$

Since Hooke's law and Cauchy's strain are both linear relationships, successive deformations may simply be added together.
The relationship between Young's modulus, Poisson's ratio and lame' coefficients are obtained from $Eq.1, Eq.2$ and$Eq.3$

$$p = (2\mu + \lambda)\frac{p}{\in} - 2\lambda V\frac{p}{\in}$$
$$0 = -(2\mu + \lambda)V\frac{p}{\in} + \lambda(-V + 1)\frac{p}{\in}$$

Solving for $\in$ and V, we obtain,

$$\in = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

and

$$V = \frac{\lambda}{2(\lambda + \mu)}$$

Conversely, we may express the lame' coefficients in terms of young's modulus and Poisson's ratio.

$$\lambda = \frac{\in V}{(1 - 2V)(1 + V)}$$

and

$$\mu = \frac{\in}{2(1 + V)}$$

Equations of Motion

Let $u(x,t)$ be the displacement of x at time t.

$$u = (u_1, u_2, u_3)^T$$

Using Newton's second law of motion, we will have:

$$\rho\frac{\partial^2 u_j}{\partial t^2} - \nabla\sigma_j = 0 \quad , \qquad (5)$$

Or

$$\rho\frac{\partial^2 u_j}{\partial t^2} - \sum_{k=1}^{3}\frac{\partial\sigma_{jk}}{\partial x_k} = 0 \ ,$$

where $j = 1,2,3.$ ρ *here* is the density of the body.The strain tensor isobtained from Hooke's law; It follows from

$$\sigma = 2\mu\varepsilon + \lambda T_r(\varepsilon)I \ . \quad (6)$$

I here represent the identity matrix. We can replace ε in equation (6). By substitution, we will have:

$$\sigma_{ij} = \lambda\left(\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Or

$$\sigma_{ij} = \lambda(\nabla\cdot u)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (7)$$

By taking the second partial derivative of equation (7), we will have:

$$\sum_{j=1}^{3}\frac{\partial \sigma_j}{\partial x_j}=\lambda\sum_{j=1}^{3}\left[\frac{\partial(\nabla\cdot u)\sigma_{ij}}{\partial x_j}\right]+\mu\sum_{j=1}^{3}\left(\frac{\partial^2 u_i}{\partial {x^2}_j}+\frac{\partial^2 u_j}{\partial x_i\partial x_j}\right)$$

or

$\sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j}=\lambda\frac{\partial(\nabla\cdot u)}{\partial x_i}+\mu\left(\nabla^2 u_i+\frac{\partial(\nabla\cdot u)}{\partial x_i}\right)$(8)

By using Newton's second law of motion, that is,

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j}=0,$$

We will have;

$$\rho\frac{\partial^2 u_i}{\partial t^2}-(\lambda+\mu)\frac{\partial(\nabla\cdot u)}{\partial x_i}-\mu\nabla^2 u_i=0$$

Clearly;

$$\rho\frac{\partial^2 u}{\partial t^2}-(\lambda+\mu)\,\nabla(\nabla\cdot u)-\mu\nabla^2 u=0 \qquad (9)$$

The above equation is **Naiver equation for an elastic body.**

Helmholtz decomposition

This theorem states that, "any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl -free) vector field and a solenoidal (divergence -free) vector field".

We can use the Helmholtz decomposition to split the vector field u into a sum of an irrotational (curl-free) vector field u_L and a solenoidal (divergence -free) vector fieldu_S.

$$u=u_L+u_S \qquad (10)$$

Infinitesimal Brief Proof based on distributions

Let $\delta(x-y)$ be the Dirac-Delta function at in $x\in\mathbb{R}^3$. We have,

$$\delta(x-y)=\frac{-1}{4\pi}\nabla^2\left(\frac{1}{|x-y|}\right)$$

and

$$V(x)=\langle F(y),\delta(x-y\ \rangle$$

$$=\frac{-1}{4\pi}\nabla^2\,\langle F(y),\frac{1}{|x-y|}\rangle$$

Now use the identity: for any vector field $w(x)$, we have,

$$\nabla^2 w=\nabla(\nabla\cdot w)-\nabla\times(\nabla\times w)$$

So that,

$$V(x)=A(x)+B(x)$$

With,

$$A(x)=\frac{-1}{4\pi}\nabla\left(\nabla\cdot\langle F(y),\frac{1}{|x-y|}\rangle\right)$$

$$B(x)=\frac{-1}{4\pi}\nabla\times\left(\nabla\times\langle F(y),\frac{1}{|x-y|}\rangle\right)$$

A is curl-free since the curl of a solenoidal is always 0. B is divergence-free since the divergence of a curl is always 0.

with

$$\begin{pmatrix}\nabla\times u_L=0\\ \nabla.u_S=0\end{pmatrix}(11)$$

$$\nabla\times u_L=0$$

is the (curl-free condition).

u_L is the longitudinal motion which gives the primary wave and u_S is the shear motion which gives the shear wave.

It follows from (9) ,(11) and $u_L,u_S=0$ at infinity that gives these equations:

$$\rho\frac{\partial^2 u_L}{\partial t^2} - (\lambda + 2\mu)\nabla^2 u_L = 0$$
$$\rho\frac{\partial^2 u_S}{\partial t^2} - \mu\nabla^2 u_S = 0$$

Let

$$C_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

and

$$C_S = \sqrt{\frac{\mu}{\rho}}$$

So that,

$$\frac{\partial^2 u_L}{\partial t^2} - C^2{}_L\nabla^2 u_L = 0$$

and

$$\frac{\partial^2 u_S}{\partial t^2} - C^2{}_S\nabla^2 u_S = 0 \qquad (12)$$

C_L *and* C_S are the waves speed. μ here representsthe shear modulus of the soil that can be obtained from the direct shear test.

Since $C_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$

and

$$C_S = \sqrt{\frac{\mu}{\rho}}$$

We can make the substitution in C_L and C_S. Making this substitution, we will have:

$$C_L = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}, \quad C_S = \sqrt{\frac{E}{2\rho(1+\nu)}}$$

The ratio of two speed is denoted by k; $k = \frac{C_L}{C_S}$. The ratio of the two speed (k) can also be expressed as:

$$k = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2(1-\nu)}{1+2\nu}} \qquad 0 \le \nu \le 0.5$$

v here represents the poison's ratio. It is a soil parameter. It is worth mentioning that,
$k > 1,$.

this means that the longitudinal speed is greater than the shear speed

HARMONIC WAVES

A harmonic wave is a wave with a frequency that is a positive integer multiple of the frequency of the original wave, known as the fundamental frequency.

Suppose that the longitudinal wave u_L is harmonic wave propagating in the x_1, x_2 plane in the direction of vector

$$\vec{n} = \begin{pmatrix} \cos\propto \\ \sin\propto \\ 0 \end{pmatrix}$$

Set

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

We will have;

$$u_L(x,t) = u_L\,\vec{n}\, e^{ik_L(\vec{n}\cdot\vec{x} - C_L t)} = m_L\,\vec{n}\, e^{i(k_L\vec{n}\cdot\vec{x} - wt)}$$

where

m_L: *is the* complex amplitude
k_L: *is the* wave number,
$w = k_L C_L$ is the frequency
C_L: *is the* wave speed

$\overrightarrow{k_L} = k_L\, \vec{n}$: is the wave vector

The shear harmonic wave $u_S(x,t)$ has two components. They are;
(a) Horizontal component, that is the component along the vector

$$\vec{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and (b)Vertical component, which is the component along the vector

$$\vec{z} \times \vec{n} = \begin{pmatrix} -\sin\propto \\ \cos\propto \\ 0 \end{pmatrix}$$

So, the shear harmonic wave

$$u_S(x,t) = M_V\ \vec{z}\ \times \vec{n}\ \ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)} + M_H\ \vec{z}\ \ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)} (13)$$

where M_V *and* M_H are the complex wave amplitudes C_S is the wave speed;

$$C_S = \frac{w}{k_S} \quad \textit{where w here represent the frequency}.$$

We can verify that $u_S(x,t)$ is divergence free.
Here, the divergence is only with respect to the space variables in the X vector.

$$u_S(x,t) = \vec{z}\ \times \vec{n}\ .\ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)} + M_H.\ \ \vec{z}\ .\ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

$$\vec{x} = (x_1, x_2,.\ xn), \vec{n} = (n_1, n_2 \ldots n_N)$$

$$\frac{\partial k_S}{\partial x_i} = M_V.\ (\vec{z}\ \times \vec{n}\ ).\ ik_S.n_i.e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)} + M_H.\vec{z}\ .\ ik_S.n_i.e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

Divergence (u_S)

$$= \sum_{j=1}^{N} \frac{\partial U_S}{\partial x_i} = \sum_{i=1}^{n} [\ M_V.\ (\vec{z}\ \times \vec{n}\ ).\ ik_S.n_i].\ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

$$= [\ i.\ k_S.M_V[\vec{z}\ \ \times \vec{n}\ ).\vec{n}\ ] + M_H.\ i.\ k_S\ (\vec{z}\ .\vec{n}\ )].e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

$$(\vec{z}\ \times\ \vec{n}\ ).\ \vec{n} = 0, \vec{z}\ .\vec{n} = 0$$

$$= 0.$$

This can be seen that the divergence is 0. This means that $u_S(x,t)$ is divergence free.
Consider u as a vector

$$u(x,t) = u_L(x,t) + u_S(x,t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

These vectors have components

$$u_1(x,t) = \ u_L \cos\propto e^{ik_S(\vec{n}\cdot\vec{x} - C_L t)} - \ u_v \sin\propto e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

$$u_2(x,t) = \ u_L \sin\propto e^{ik_L(\vec{n}\cdot\vec{x} - C_L t)} + \ m_v \cos\propto e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

$$u_3(x,t) = u_H\ \vec{z}\ \ e^{ik_S(\vec{n}\cdot\vec{x} - C_S t)}$$

It is worth mentioning that u_3 is decoupled and consists only of the shear horizontal wave.

RAYLEIGH WAVES

The ground is at $x_2 = 0$, and $x_2 \leq 0$.
The wave propagates on the earth when $x_2 \leq 0$.

$$u(x_1, x_2, t) = u_L(x_1, x_2, t) + u_S(x_1, x_2, t)$$
$$= \big(L(x_2) + S(x_2)\big)e^{i(kx_1 - wt)}$$

Where; $L(x_2)$ *and* $S(x_2)$ are vector-valued functions. Since u_L and u_S satisfy the wave equations;

$$\frac{\partial^2 u_L}{\partial t^2} - C^2{}_L \nabla^2 u_L = 0$$

and

$$\frac{\partial^2 u_S}{\partial t^2} - C^2{}_S \nabla^2 u_S = 0$$

It follows from (9). Then L(x_2) and S(x_2) satisfy the equations

$$L'' + \left(\frac{w^2}{C^2{}_L} - k^2\right) L = 0$$

and

$$S'' + \left(\frac{w^2}{C^2{}_S} - k^2\right) S = 0$$

The amplitude deceases, therefore,

$$\left(\frac{w^2}{C^2{}_L} - k^2\right) < 0$$

and

$$\left(\frac{w^2}{C^2{}_S} - k^2\right) < 0,$$

set

$$\frac{w^2}{C^2{}_L} - k^2 = -k_L{}^2$$

and

$$\frac{w^2}{C^2{}_S} - k^2 = -k^2{}_s$$

$k_L > 0, k_S$>0. Making the substitution we will have:

$$L'' - k^2{}_L\, L = 0 \text{ and } S'' - k^2{}_S S = 0 (14)$$

solving this second order differential equations with constant coefficients, we will have:

$$L(x_2) = Me^{k_L x_2} + Ne^{-k_L x_2}$$
$$S(x_2) = Pe^{k_S x_2} + Qe^{-k_S x_2}$$

Since $u \rightarrow 0$ *and* $x_2 \rightarrow -\infty$ *then*, $N = Q = 0$.

$L(x_2) = M\boldsymbol{e}^{\boldsymbol{k_L x_2}}, S(x_2) = P\boldsymbol{e}^{\boldsymbol{k_S x_2}}$ *with* $N, P \in \mathbb{R}^3$ *are constants*.

It follows that;

$$u(x_1, x_2, t) = (Me^{k_L x_2} + Pe^{k_S x_2})e^{i(kx_1 - wt)} (15)$$

Note that since $e^{k_L x_2}$ *and* $e^{k_S x_2}$ *approaches* 0 *as* x_2 *approaches* $-\infty$, then the wave will be only felt on the surface $x_2$near boundary$x_2 = 0$.

The longitudinal component $Me^{k_L x_2} e^{i(kx_1 - wt)}$ must be curl-free and $Pe^{k_S x_2} e^{i(kx_1 - wt)}$ must be divergence-free.

So,

$$\nabla x \left(M\boldsymbol{e}^{\boldsymbol{k_L x_2}} \boldsymbol{e}^{\boldsymbol{i(kx_1 - wt)}}\right) = 0$$

and

$$\nabla \cdot \left(P\boldsymbol{e}^{\boldsymbol{k_S x_2}} \boldsymbol{e}^{\boldsymbol{i(kx_1 - wt)}}\right) = 0$$

If

$$M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$$

and

$$P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Then; the curl − free condition implies; $M_3 k_L = 0, M_3 ik = 0, ikM_2 - k_L M_1 = 0$

So,

$$M_3 = 0, \text{and } M_2 = \frac{k_L}{ik} M_1$$

The divergence − free condition gives; $ikP_1 + k_S P_2 = 0$, we can solve for P_2

$$P_2 = -\frac{ik}{k_S} P_1 \text{ since } u \equiv 0 \text{ for } t \rightarrow \infty, \text{then } P_3 = 0$$

We have then;

$$u(x_1,x_2,t)=\left[\begin{pmatrix} M_1 \\ \frac{k_L}{ik}M_1 \\ 0 \end{pmatrix} e^{k_L x_2}+\begin{pmatrix} P_1 \\ -\frac{ik}{k_S}P_1 \\ 0 \end{pmatrix} e^{k_S x_2}\right] e^{i(kx_1-wt)}$$

Set $a=M_1$ and $b=P_1$. This gives:

$$u(x_1,x_2,t)=\left[\begin{pmatrix} 1 \\ \frac{k_L}{ik} \\ 0 \end{pmatrix} ae^{k_L x_2}+\begin{pmatrix} 1 \\ -\frac{ik}{k_S} \\ 0 \end{pmatrix} be^{k_S x_2}\right] e^{i(kx_1-wt)}$$

Impose free boundary condition as $x_2=0$. This means that there is no stress.

$$\sigma_{12}\Big|_{x2=0}=0 \Rightarrow \frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big|_{x2=0}=0$$

$$\sigma_{22}\Big|_{x2=0}=0 \Rightarrow \left[(\lambda+2\mu)\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial x_2}\right)-2\mu\frac{\partial u_1}{\partial x_1}\right]\Big|_{x2=0}=0$$

The first condition implies that;

$$(2k_L k_S)a=(k^2{}_S+k^2)b=0(16)$$

The second condition gives;

$$(\lambda+2\mu)\left(ika+ikb+\frac{k^2{}_L}{ik}a-ikb\right)-2\mu(ika+ikb)=0(17)$$

Let

$$k=\sqrt{\frac{\lambda+2\mu}{\mu}}.$$

Making the substitution, we will have:

$$k^2(k^2{}_L+k^2)a+2k^2(a+b)=0$$

The system becomes:

$$(2k_L k_S)a+(k^2{}_s+k^2)b=0$$
$$k^2(k^2{}_L-k^2)a+2k^2(a+b)=0$$

Since,

$$\frac{w^2}{C^2{}_L}-k^2=-k^2{}_L\,;\ \frac{w^2}{C^2{}_s}-k^2=-k^2{}_s,\qquad k=\frac{C_L}{C_S}$$

Then,

$$w^2=c_L{}^2(k^2-k^2{}_L),\ \ w^2=C^2{}_s(k^2-k^2{}_s),\ k^2=\frac{k^2-k^2{}_s}{k^2-k^2{}_L}$$

Clearly,

$$(2k_L k_S)a+(k^2{}_s+k^2)b=0$$

$$\left[\frac{k^2-k^2{}_s}{k^2-k^2{}_L}(k^2{}_L-k^2)a+2k^2a\right]+2k^2b=0$$

Multiply the above expression in the bracket by $(k^2-k^2{}_L)$. We now have:

$$(2k_L k_S)a+(k^2{}_s+k^2)b=0$$
$$(h^2+k^2{}_s)a+2k^2b=0$$

In order for the systems to have a nontrivial solution, the determinant "$Det=0$"

$$(4k^2k_L k_S)-(k^2+k^2{}_s)^2=0$$

Now, we can replace,

$$k_L{}^2=k^2-\frac{w^2}{C^2{}_L}\ and\ \ k_S{}^2=k^2-\frac{w^2}{C^2{}_s}$$

This gives,

$$4k^2\sqrt{k^2-\frac{w^2}{C^2{}_L}}\sqrt{k^2-\frac{w^2}{C^2{}_S}}-\left(2k^2-\frac{w^2}{C^2{}_S}\right)^2=0 \qquad (18)$$

Squaring both side of the equation, we will have:

$$16k^4\left(k^2-\frac{w^2}{C^2{}_L}\right)\left(k^2-\frac{w^2}{C^2{}_S}\right)-\left(2k^2-\frac{w^2}{C^2{}_S}\right)^4=0$$

To simplify this expression further, let

$$X=\frac{\boldsymbol{w}}{\boldsymbol{C_S k}}$$

$$16k^8\left(1-\frac{w^2}{C^2{}_L k^2}\right)(1-X^2)-k^8(2-X^2)^4=0$$

Since

$$k=\frac{C_L}{C_S},\ \frac{w^2}{C^2{}_L k^2}=\frac{C^2{}_S}{C^2{}_L},\frac{w^2}{C^2{}_S k^2}=\frac{X^2}{k^2},$$

our new equation becomes:

$$16\left(1-\frac{X^2}{k^2}\right)(1-X^2)-(2-X^2)^4=0$$

The binomial expansion of the term

$$(2-X^2)^4 \text{ is } X^8-8X^6+24X^4-32X^2+16.$$

For the term

$$\left(1-\frac{X^2}{k^2}\right)(1-X^2),$$

we will have:

$$1-X^2-\frac{X^2}{k^2}+\frac{X^4}{k^2}.$$

Now, our new equation becomes:

$$16\left(1-X^2-\frac{X^2}{k^2}+\frac{X^4}{k^2}\right)-(X^8-8X^6+24X^4-32X^2+16)=0$$

By reducing the above equation, we will have:

$$X^8-8X^6+8\left(3-\frac{2}{k^2}\right)X^4+16\left(1-\frac{1}{k^2}\right)X^2=0$$

$$X^6-8X^4+8\left(3-\frac{2}{k^2}\right)X^2+16\left(1-\frac{1}{k^2}\right)=0. \qquad (19)$$

This is the Rayleigh wave equation. It is a 3rd degree equation in X^2.
Since for a given material,

$$k=\sqrt{\frac{2(1-v)}{1-2v}}=\frac{C_L}{C_S}$$

depends on the properties of the materials. The roots of (18) implies the possible values of

$$X=\frac{w}{C_S k}.$$

However, not all such roots are admissible. Since

$$\frac{w^2}{C^2{}_L}-k^2=-K^2{}_L<0,\ \frac{w^2}{C^2{}_S}=-K^2{}_S<0,$$

this implies that,

$$-K^2{}_L=X^2\cdot C^2{}_S k^2/C^2{}_L-k^2<0\to X^2/k^2-1<0$$

$$-K^2{}_S=X^2k^2\text{-}k^2<0,$$

this implies:

$$X^2-1<0$$

That is,

$$(X<1,\qquad X<k)$$

Since

$$C_L > C_S \quad , \quad k > 1$$

We need to prove that the cubic equation has a single solution in the interval (0, 1) for any ratio k.

$$Y^3 - 8Y^2 + 8\left(3 - \frac{2}{k^2}\right)Y + 16\left(1 - \frac{1}{k^2}\right) = 0$$

Has only one solution in their interval $(0, 1)$. We reached the following equation;

$$Y^3 - 8Y^2 + 8\left(3 - \frac{2}{k^2}\right)Y - 16\left(1 - \frac{1}{k^2}\right) = 0$$

We note that $k = \frac{C_L}{C_S}$ is always > 1 since the longitudinal speed wave in > shear speed wave.
We need to show that the Rayleigh waves equation derived above has a unique solution $Y \in (0, 1)$ for every $k \geq 1$

Let the function f(y) equals the polynomial equation, that is,

$$f(Y) = Y^3 - 8Y^2 + 8\left(3 - \frac{2}{k^2}\right)Y - 16\left(1 - \frac{1}{k^2}\right)$$

$$f(0) = -16\left(1 - \frac{1}{k^2}\right) < 0; \ \ f(1) = 1 - 8 + 24 - \frac{16}{k^2} - 16 + \frac{16}{k^2} = 1 > 0$$

So, $f(Y) = 0$ has at least one solution in $(0, 1)$. We need to show that there are no others.

$$f'(Y) = 3\,Y^3 - 16Y + 8\left(3 - \frac{2}{k^2}\right);$$

The roots of $f'(Y) = 0$ are;

$$Y^{\pm} = \frac{8 \pm \sqrt{64 - 24\left(3 - \frac{2}{k^2}\right)}}{3} = \frac{8}{3} \pm \frac{\sqrt{\frac{48}{k^2} - 8}}{3}$$

$$Y^{\pm} = \frac{8}{3} \pm \frac{\sqrt{8\left(\frac{6}{k^2} - 1\right)}}{3}$$

If $\frac{6}{k^2} - 1 < 0, \ i.e. k^2 > 6, \ k > \sqrt{6}$

then, $f'(Y)$ has no real roots, $f'(Y) > 0 \ for\ every\ y.\ f$ is increasing.Equation $f(Y) = 0$ has only one solution.

Note that,

$$Y^+ = \frac{8}{3} + \frac{\sqrt{8\left(\frac{6}{k^2} - 1\right)}}{3} > 1,$$

for every k,

$$k \leq \sqrt{6}$$

Consider two cases depending on whether $Y^- < 1$ or not. Where

$$Y^- = \frac{8}{3} - \frac{\sqrt{8\left(\frac{6}{k^2} - 1\right)}}{3}$$

<u>Case 1:</u>
$Y^- \geq 1$. This occurs if,

$$\sqrt{\frac{48}{33}} < k < \sqrt{6}\ ;\ 1.21 < k < 2.45$$

Since $f'(Y) > 0$ for $Y < Y^-$, then f is increasing at the interval $(0, 1) \subset (0, Y^-)$ and so $f(Y) = 0$ has only root in (0, 1).

<u>Case 2</u>: (Note that if $k > 1$, then $Y^- > 0$)

Suppose $0 < Y^- < 1$, the function f has a local maximum at Y^- and a local minimum at Y^+ with $0 < Y^- < 1 < Y^+$

Since $f(1) = 1, then\ f(Y^-) > f(1) = 1$, then f is increasing $(0,\ Y^-)$ and decreasing on $(Y^-,\ 1)$

$f(y) = 0$ has a single solution on $(0, \quad Y^-)$ andno solution on $(Y^-, \quad 1)$

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